

# VERMA MODULES, EXTREMAL VECTORS, AND SINGULAR VECTORS ON THE NON-CRITICAL $N=2$ STRING WORLDSHEET

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We formulate the general construction for singular vectors in Verma modules of the affine  $sl(2|1)$  superalgebra. We then construct  $sl(2|1)$  representations out of the fields of the non-critical  $N=2$  string. This allows us to extend naturally to  $sl(2|1)$  several crucial properties of the  $N=2$  superconformal algebra, first of all the construction of extremal states (an analogue of different pictures for non-free fermions) and the spectral flow transform (which then affects the Liouville sector). We further evaluate the affine  $sl(2|1)$  singular vectors in the realization of  $sl(2|1)$  provided by the  $N=2$  string. We establish that, with a notable exception, the respective singular vectors are in a  $2:1$  correspondence, namely two different  $sl(2|1)$  singular vectors evaluate as an  $N=2$  superconformal singular vector (however, those singular vectors that are labelled by a pair of positive integers get these integers transposed under the reduction). We also analyse the ‘exceptional’ cases, which amount to a series of  $sl(2|1)$  singular vectors, labelled by  $r \geq 1$ , which do not have an  $N=2$  counterpart, and discuss the mechanism by which the multiplicity of singular vectors becomes equal to two at certain points in the weight spaces of both algebras.

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# 1 Introduction

String theories, taken in their worldsheet formulation, are known to possess a number of hidden worldsheet symmetries [36, 15, 17, 46, 60] and a series of embeddings [13, 31, 56, 7, 12, 14, 58]. One is particularly interested [60] in relations of string theories to the affine Lie algebras (such relations can be observed at different levels, see e.g., [2, 38, 26]). At the same time, a series of embeddings found between string theories show that lower-supersymmetric strings can be ‘prepared’ as some special states of the higher-supersymmetric ones.

The  $N=2$  strings [1, 51, 32, 54, 57, 14, 47], whose role in the M-theory has recently been proposed in [45, 52] (see also [43]), are interesting also because the non-critical  $N=2$  string provides a direct realization of the affine  $sl(2|1)$  superalgebra on the worldsheet in the conformal gauge [64]. One can therefore expect that a number of properties of the affine  $sl(2|1)$  algebra would be seen in  $N=2$  strings. The most far-reaching consequences would be those concerning the structure of physical states (the BRST cohomology), aimed at an  $N=2$  extension of the results of [49] (see also [19]), and the fusion rules. As a step towards that aim one has to consider first the structure of the  $sl(2|1)$  representations realized in the non-critical  $N=2$  string, and this is one of the problems that we address in the present paper.

As is the case with the much better studied  $N=0$  models, one is particularly interested in those representations whose highest-weight states allow for the existence of singular vectors [10, 24, 50, 40]; then, after factoring out the submodule generated by the singular vectors, one is left, in the Virasoro and affine  $sl(2)$  cases, with an irreducible representation. As to the *rank-3* affine  $sl(2|1)$  algebra, the situation can be more complicated due to the presence of *subsingular* vectors (i.e., the module obtained by factoring over ‘level-1’ singular vectors may not be irreducible). There has been a constant interest in the structure and explicit constructions of singular vectors of the various infinite-dimensional algebras [50, 11, 8, 9, 35, 22, 68, 6, 18, 23]. In particular, relations between different algebras, such as, primarily, the Hamiltonian reduction, have in some cases been shown to extend to the respective singular vectors [35]. The appearance of the  $sl(2|1)$  algebra in the non-critical  $N=2$  string is significant since it is by the Hamiltonian reduction of the affine  $sl(2|1)$  that one can obtain the  $N=2$  superconformal algebra [16, 15, 39]. Thus the construction of the affine  $sl(2|1)$  out of  $N=2$  superconformal matter ‘dressed’ with some free fields (the ghosts and the Liouville) can be considered as an ‘inversion’ of the Hamiltonian reduction [63], and one may expect that these two algebras would have ‘related’ representations and ‘related’ singular vectors<sup>1</sup>. From the correspondence between singular vectors, a relation between the ‘Lian–Zuckerman’ states of the respective theories should ultimately follow, since the Lian–Zuckerman states are related to singular vectors in the Verma modules (even though the explicit form of that relation has not been worked out in sufficient generality, see [41, 55]). A similar correspondence should also exist then between the fusion rules.

While neither the ‘Lian–Zuckerman’ states for  $N=2$  strings nor the fusion rules for  $sl(2|1)$  are yet

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<sup>1</sup>That the Hamiltonian reduction tends to apply nicely to singular vectors does not follow from the first-principles; indeed, the Hamiltonian reduction does in general destroy even the Lie-algebra structure (recall the reductions of ‘higher’-rank Kač–Moody algebras, resulting in W-, not Lie, algebras), and there seem to be no general grounds to expect that it would induce any reasonable relation between representations. In our opinion, the possibility to ‘extend’ the Hamiltonian reduction to singular vectors has rather to do with the existence of the ‘inversion’ of the reduction [63]. This raises, however, an intriguing problem of ‘inverting’ the  $sl(3) \rightarrow W_3$  Hamiltonian reduction.

known, singular vectors are a tool of major importance in the analysis of representations of these algebras.<sup>2</sup> In this paper, firstly, we give the general construction for singular vectors of the affine  $s\ell(2|1)$  algebra. In doing so, we will employ the properties of the  $s\ell(2|1)$  algebra that turn out to have analogues in the case of the  $N=2$  superconformal algebra. An important feature of the affine  $s\ell(2|1)$  algebra, which shows up in its representation theory, is the existence of a spectral flow transform, similar (and in fact, closely related) to the one which is known to be important in the  $N=2$  superconformal algebra [61, 48, 67, 34] and  $N=2$  (critical) strings [14, 47, 44]. We will show that the spectral flow transform extends naturally from the  $N=2$  ‘matter’ to the  $s\ell(2|1)$  algebra and therefore to the noncritical  $N=2$  string, where it also affects the Liouville superpartner. Secondly, we will reduce the problem of the analysis of  $s\ell(2|1)$  singular vectors to the  $N=2$  superconformal algebra: by evaluating  $s\ell(2|1)$  singular vectors in the realization provided by the  $N=2$  string, we will show that the  $s\ell(2|1)$  singular vectors are in a  $2:1$  correspondence with the  $N=2$  singular vectors, except for a series of  $s\ell(2|1)$  singular vectors, labelled by positive integers, which do not correspond to any  $N=2$  singular vector (which suggests, in particular, that the  $s\ell(2|1)$  fusion rules would have an extra series as compared with the  $N=2$  fusion rules). We will also find that those  $s\ell(2|1)$  singular vectors which, in some standard nomenclature, are labelled by a pair  $(r, s)$  of positive integers, map to the  $(s, r)$  singular vectors of the  $N=2$  algebra. At the same time, none of the singular vectors are ‘lost’ in the realization of the affine  $s\ell(2|1)$  provided by the non-critical  $N=2$  string: in terms of the fields of the  $N=2$  string, none of the singular vectors vanishes, and different singular vectors evaluate differently. All this will be done for an arbitrary (complex) level  $k \neq -1$ , and, accordingly, an arbitrary  $N=2$  central charge  $c \neq 3$ . The general construction of  $s\ell(2|1)$  singular vectors is then to be applied to the case of rational  $k$  similarly to how this is done for the ordinary MFF construction [50]: for the  $s\ell(2)$  case, for example, each of the two MFF formulae produces a singular vector as many times as there are different solutions to the equations on the parameters (in that case, the spin and the level) that guarantee the existence of a singular vector.

Another algebra which emerges on the  $N=2$  non-critical string worldsheet is the  $N=4$  superconformal algebra [17]. In that respect, the  $N=2$  string repeats, with some interesting modifications, the relations existing around the non-critical bosonic string:

$$\begin{array}{ccc}
 s\ell(2) & & N=2 \\
 \swarrow \quad \searrow & & \nearrow \\
 & \text{Virasoro} &
 \end{array}
 \tag{1.1}$$

where the  $N=2$  superconformal algebra is realized precisely by adding to the Virasoro algebra the free fields of the non-critical string in the conformal gauge. On the other hand, reconstructing the affine  $s\ell(2)$  currents takes yet another free boson [62] (the downward arrow being the Hamiltonian reduction). The main variation that we have in the  $N=2$  string, apart from the  $(N=0) \rightarrow (N=2)$  ‘translation’, is that the  $s\ell(2|1)$  and  $N=4$  algebras are realized on the *same* space of fields, which means that the upward arrows denote ‘dressing’ of the  $N=2$  algebra with the same collection of free fields in both cases, namely

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<sup>2</sup>Another, seemingly unrelated, reason to be interested in singular vectors comes from integrable ‘massive’ models [5, 53].

with the Liouville and ghost multiplets of the  $N=2$  string:

$$\begin{array}{ccc}
 sl(2|1) & & N=4 \\
 & \swarrow \quad \searrow & \\
 & N=2 \text{ 'matter'} &
 \end{array}
 \tag{1.2}$$

It should be recalled that in (1.1), a relation between, in that case, the affine  $sl(2)$  and  $N=2$  singular vectors exists, but is far from trivial: a subclass of  $N=2$  singular vectors are isomorphic to the  $sl(2)$  singular vectors [62, 30], while the ‘bulk’ of  $N=2$  singular vectors are related to singular vectors in  $sl(2)$  modules that are not of the usual highest-weight type, but rather have infinitely many equivalent ‘almost-highest-weight’ vectors. The  $N=2$  counterpart of these states are the *extremal* vectors [29].

On the other hand, in the diagram (1.2), where the  $N=2$  algebra is a ‘primitive’ ingredient, little is known, beyond [17], about its  $N=4$  side<sup>3</sup>. As to the  $sl(2|1)$  algebra, we will show that it combines, in a rather non-trivial way, certain properties of the  $sl(2)$  and  $N=2$  algebras. In particular, the extremal states of the  $N=2$  algebra lift naturally to the larger algebras. This leads to very suggestive similarities between the theories of  $sl(2|1)$  and  $N=2$  singular vectors, both being closely related to the respective extremal vectors. The analysis of extremal vectors does immediately produce certain series of singular vectors, by a kind of ‘multiplet-shortening’ mechanism, and is actually very suggestive as to how the general construction for singular vectors can be built. The latter takes introducing the ‘continued’ extremal vectors and, accordingly, the continued operators that generate them from the vacuum. These operators are realized in terms of ‘continued products’ of fermions.

Thus the idea to consider extremal vectors [29], elaborated in application to the  $N=2$  superconformal algebra in collaboration with I. Tipunin [67], turns out to be very efficient for the affine  $sl(2|1)$  as well (and looks quite promising also for any algebra with at least two fermionic currents, in fact with a spectral flow transform). It may be hoped that this analysis will be useful to relate the  $sl(2|1)$  and  $N=2$  fusion rules (for the related material, see [4, 3, 59, 34]).

The extremal states can be viewed as a generalization of different *pictures* [33] to the case of non-free fermions. Recall that for the free first-order bosonic systems, different pictures are inequivalent in the sense of Verma modules, while for free fermions, on the contrary, they *are* equivalent. For the interacting fermions, the situation turns out to be somewhat ‘intermediate’: generically, the extremal states are still equivalent to each other, but it may (and does) happen for some values of the relevant parameters (the weights) that the equivalence breaks down. To continue with the analogy with the first-order free bosons and fermions, recall that the pictures in the bosonic case are changed by the exponential of a current that participates in ‘bosonizing’ the system,  $\exp \phi$ . By considering operators like  $\exp \alpha \phi$ , we can change the picture arbitrarily (at least in principle, at the expense of non-localities). The same is true for bosonized free fermionic system. However, when the fermions are non-free, such a bosonization no longer exists, and changing the picture by an arbitrary number is realized by the ‘continued’ products of modes of the fermionic generators. For the  $N=2$  superconformal algebra, for example, we have two fermionic currents

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<sup>3</sup>An intriguing point, however, is that the construction of  $sl(2|1)$  singular vectors that we will consider below is also similar to a construction of certain  $N=4$  singular vectors, even though the latter has been realized only in a very special case [65].

$\mathcal{Q}_m$  and  $\mathcal{G}_n$ , with

$$\{\mathcal{G}_m, \mathcal{Q}_n\} = 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\epsilon}{3}(m^2 + m)\delta_{m+n,0},$$

and the ‘continued’ operators  $q(a, b)$  and  $g(a, b)$  can heuristically be thought of as  $\prod_a^b \mathcal{Q}_\mu$  and  $\prod_a^b \mathcal{G}_\mu$  respectively. Had  $\mathcal{Q}$  and  $\mathcal{G}$  been free and bosonized through a free scalar, these ‘continued’ operators would have been constructed ‘explicitly’. When no such bosonization is possible, however, the set of bosonization rules can nevertheless be replaced by a set of *algebraic rules* to deal with the new operators. For instance, when considering the commutator of the Virasoro generators  $\mathcal{L}_n$  with  $g(a, b)$ , we can first take  $a$  and  $b$  such that  $b - a = N$  is a positive integer, then we observe that in the commutator

$$[\mathcal{L}_1, \mathcal{G}_a \dots \mathcal{G}_{a+N}]$$

only the commutator  $[\mathcal{L}_1, \mathcal{G}_{a+N}]$  gives a non-zero contribution; for  $[\mathcal{L}_2, \mathcal{G}_a \dots \mathcal{G}_{a+N}]$ , similarly, only  $\mathcal{G}_{a+N-1}$  and  $\mathcal{G}_{a+N}$  would contribute non-vanishing results, and so forth. This therefore extends to *arbitrary complex*  $a$  and  $b$  as [66]

$$[\mathcal{L}_n, g(a, b)] = \sum_{i=0}^{n-1} g(a, b - i - 1) [\mathcal{L}_n, \mathcal{G}_{b-i}] \mathcal{G}_{b-i+1} \dots \mathcal{G}_b, \quad n \geq 1$$

(for negative  $n$ , one counts from the left-hand end).

These rules can be developed into a consistent algebraic scheme, which does in a sense play the role of bosonization of non-free fermions. It has appeared in [66], while the states that these operators create from the vacuum have been introduced, and their representation-theoretic importance was stressed, in [29]; we take over from [29] the name *extremal* for these states. Let us also mention that the role of extremal states is absolutely essential already in more ‘classical’ problems, like that of characterizing the image of the affine  $sl(2)$  highest-weight modules under the Kazama–Suzuki mapping [30].

This paper is organized as follows. In Section 2, we consider highest-weight and extremal vectors in the affine  $sl(2|1)$  Verma modules. In Section 3, we present the general construction of the  $sl(2|1)$  singular vectors. As we have remarked, the singular vectors are essentially divided onto those read off directly from the extremal vectors, and those which require a ‘continuation’. The continued part of the construction can be given in a direct analogy with the MFF recipe for bosonic algebras [50], however a nice feature of  $sl(2|1)$  as a superalgebra is the role of the fermionic currents in the construction (in particular, the MFF-like continuation can be performed entirely in terms of ‘continued’ products of the fermions). Section 4 is devoted to the analysis of representations of  $sl(2|1)$  realized on the noncritical  $N=2$  string worldsheet by tensoring the  $N=2$  superconformal matter with the corresponding free fields. An essential point here is that no assumptions are made on the nature of the ‘matter’  $N=2$  superconformal theory, in particular its currents are not supposed to be ‘bosonized’ in terms of free fields; in that sense, the realization of  $sl(2|1)$  we are going to consider is *not* a free-field realization. This will be important in Section 5, where we consider how the  $sl(2|1)$  singular vectors from Section 3 evaluate in this realization. Again in contrast with free-field realizations, none of the  $sl(2|1)$  singular vectors vanishes; in fact, they reduce precisely to singular vectors of the underlying  $N=2$  superconformal algebra, except for a particular series labelled by positive integers.

## 2 The affine $s\ell(2|1)$ algebra

### 2.1 $s\ell(2|1)$ commutators, automorphisms, and the spectral flow

The algebra consists of four bosonic currents,  $E^{12}$ ,  $H^-$ ,  $F^{12}$ ,  $H^+$ , and four fermionic ones,  $E^1$ ,  $E^2$ ,  $F^1$ , and  $F^2$ . We will quite often drop ‘affine’ when referring to this algebra. The non-vanishing commutation relations (with the brackets  $[ , ]$  denoting the *super*commutator) read

$$\begin{aligned}
[H_m^-, E_n^{12}] &= E_{m+n}^{12}, & [H_m^-, F_n^{12}] &= -F_{m+n}^{12}, \\
[E_m^{12}, F_n^{12}] &= m\delta_{m+n,0}k + 2H_{m+n}^-, & [H_m^\pm, H_n^\pm] &= \mp \frac{1}{2}m\delta_{m+n,0}k, \\
[F_m^{12}, E_n^2] &= F_{m+n}^1, & [E_m^{12}, F_n^2] &= -E_{m+n}^1, \\
[F_m^{12}, E_n^1] &= -F_{m+n}^2, & [E_m^{12}, F_n^1] &= E_{m+n}^2, \\
[H_m^\pm, E_n^1] &= \frac{1}{2}E_{m+n}^1, & [H_m^\pm, F_n^1] &= -\frac{1}{2}F_{m+n}^1, \\
[H_m^\pm, E_n^2] &= \mp \frac{1}{2}E_{m+n}^2, & [H_m^\pm, F_n^2] &= \pm \frac{1}{2}F_{m+n}^2, \\
[E_m^1, F_n^1] &= -m\delta_{m+n,0}k + H_{m+n}^+ - H_{m+n}^-, \\
[E_m^2, F_n^2] &= m\delta_{m+n,0}k + H_{m+n}^+ + H_{m+n}^-, \\
[E_m^1, E_n^2] &= E_{m+n}^{12}, & [F_m^1, F_n^2] &= F_{m+n}^{12}
\end{aligned} \tag{2.1}$$

The affine  $s\ell(2)$  subalgebra is thus generated by  $E_m^{12}$ ,  $H_m^-$ , and  $F_m^{12}$ , and it commutes with the  $U(1)$  subalgebra generated by  $H_m^+$ .

The affine  $s\ell(2|1)$  algebra admits the following order-four automorphism

$$\begin{aligned}
E^1 &\mapsto -F^1, & E^2 &\mapsto -F^2, & E^{12} &\mapsto F^{12}, \\
F^1 &\mapsto E^1, & F^2 &\mapsto E^2, & F^{12} &\mapsto E^{12}, \\
H^- &\mapsto -H^-, & H^+ &\mapsto -H^+,
\end{aligned} \tag{2.2}$$

and also an involutive automorphism:

$$\begin{aligned}
E_n^1 &\mapsto E_n^2, & E_n^2 &\mapsto E_n^1, & H_n^+ &\mapsto -H_n^+ \\
F_n^1 &\mapsto -F_n^2, & F_n^2 &\mapsto -F_n^1,
\end{aligned} \tag{2.3}$$

(the other generators remain unchanged).

Further, the spectral flow transform

$$\mathcal{U}_\theta : \begin{aligned} E_n^1 &\mapsto E_{n-\theta}^1, & E_n^2 &\mapsto E_{n+\theta}^2, & H_n^+ &\mapsto H_n^+ + k\theta\delta_{n,0} \\ F_n^1 &\mapsto F_{n+\theta}^1, & F_n^2 &\mapsto F_{n-\theta}^2, \end{aligned} \tag{2.4}$$

(where, again, the  $s\ell(2)$  subalgebra is inert) produces an isomorphic algebra for any  $\theta \in \mathbb{C}$ . In any of the thus obtained isomorphic algebras, the fermions have integer-spaced modes with the ‘offset’  $\pm\theta$ . In particular, for  $\theta = \frac{1}{2}$ , we will have the ‘Ramond’ algebra. For  $\theta \in \mathbb{Z}$ , the spectral flow transform is an automorphism of the algebra. As we will see, the transformation (2.4) is consistent with the  $N=2$  superconformal spectral flow transform [61, 48].

Let us note finally that in terms of the normal-ordered field operators  $C(z) = \sum_{n \in \mathbb{Z}} C_n z^{-n-1}$ , where  $C = (E^1, E^2, E^{12}, H^-, H^+, F^1, F^2, F^{12})$ , the Sugawara energy-momentum tensor reads

$$\begin{aligned} T_{\text{Sug}} &= \frac{1}{k+1} \left( H^- H^- - H^+ H^+ + \frac{1}{2} E^{12} F^{12} + \frac{1}{2} F^{12} E^{12} + \frac{1}{2} E^1 F^1 - \frac{1}{2} F^1 E^1 - \frac{1}{2} E^2 F^2 + \frac{1}{2} F^2 E^2 \right) \\ &= \frac{1}{k+1} \left( H^- H^- - H^+ H^+ + E^{12} F^{12} + E^1 F^1 - E^2 F^2 \right) \end{aligned} \quad (2.5)$$

## 2.2 $s\ell(2|1)$ highest-weight modules and extremal states

Consider the highest-weight conditions. They have to be imposed in such a way as not to over-determine the system of constraints. As usual, this selects essentially the positive-moded generators, with some subtleties arising with the lowest annihilators (in some formulations, these will be viewed as ‘zero modes’ of the vacuum). As to the generators  $H^+$  and  $H^-$ , the standard Heisenberg module highest-weight conditions read

$$H_{\geq 1}^+ \approx 0, \quad H_{\geq 1}^- \approx 0. \quad (2.6)$$

A characteristic feature of fermionic systems, on the other hand, is that the distinction between creation and annihilation operators can be drawn arbitrarily, recall for instance the ‘ $q$ ’-vacua of the  $bc$  systems [33]. Although the above  $E^1, E^2, F^1$  and  $F^2$  are not *free* fermions, a similar effect does take place here as well (see the Introduction). For an arbitrary  $\theta$ , we can choose a vacuum on which  $E_{\geq -\theta + \frac{1}{2}}^1$  and  $F_{\geq \theta + \frac{1}{2}}^1$  would be annihilators<sup>4</sup>. Then, the strongest conditions we can have for  $E^2$  and  $F^2$  would be  $E_{\geq \theta - \frac{1}{2}}^2 \approx 0$  and  $F_{\geq -\theta + \frac{3}{2}}^2 \approx 0$ .

Note that the fermionic annihilation conditions imply those for the bosons as

$$\begin{aligned} E_{\geq -\theta + \frac{1}{2}}^1 &\approx 0, \quad E_{\geq \theta - \frac{1}{2}}^2 \approx 0, \implies E_{\geq 0}^{12} \approx 0, \\ F_{\geq \theta + \frac{1}{2}}^1 &\approx 0, \quad F_{\geq -\theta + \frac{3}{2}}^2 \approx 0, \implies F_{\geq 2}^{12} \approx 0. \end{aligned} \quad (2.7)$$

However, the resulting highest-weight condition in the  $s\ell(2)$  subalgebra are *not* those of the standard Verma modules; on the contrary, the  $s\ell(2)$  modules with a ‘highest-weight’ state determined by the right column in (2.7) do in fact have infinitely many equivalent highest-weight vectors, and thus should not be called Verma modules, once the latter are understood to have a unique highest-weight vector.

The highest-weight conditions (2.7) (together with (2.6)) can be strengthened consistently, so as to yield the standard Verma modules for the  $s\ell(2)$  subalgebra, as

$$\begin{aligned} E_{\geq -\theta + \frac{1}{2}}^1 |p, j, k; \theta\rangle &= 0, \quad E_{\geq \theta - \frac{1}{2}}^2 |p, j, k; \theta\rangle = 0, \quad E_{\geq 0}^{12} |p, j, k; \theta\rangle = 0, \\ F_{\geq \theta + \frac{1}{2}}^1 |p, j, k; \theta\rangle &= 0, \quad F_{\geq -\theta + \frac{3}{2}}^2 |p, j, k; \theta\rangle = 0, \quad F_{\geq 1}^{12} |p, j, k; \theta\rangle = 0, \\ H_0^+ |p, j, k; \theta\rangle &= (p - k\theta) |p, j, k; \theta\rangle, \quad H_0^- |p, j, k; \theta\rangle = j |p, j, k; \theta\rangle. \end{aligned} \quad (2.8)$$

These will be called the generalized highest-weight conditions, termed ‘generalized’ for the presence of the  $\theta$  parameter. The ‘ordinary’ highest-weight vector will be denoted as  $|p, j, k\rangle \equiv |p, j, k; 0\rangle$ ; as we are going to see, the ‘standard’ choice of  $\theta = 0$  is merely a convention:

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<sup>4</sup>Note that Eqs. (2.3) allow us to change the roles of the two pairs (<sup>1</sup> and <sup>2</sup>) of the fermionic generators.

**Theorem 2.1** *Unless  $-p \pm j + \frac{k}{2} - r(k+1) = 0$ ,  $r \in \mathbb{Z}$ , the highest-weight conditions (2.8) are equivalent for all  $\theta \in \mathbb{Z}$ .*

Indeed, applying the fermionic modes to a chosen highest-weight state, we can reach all the other integer-spaced highest-weight states. Define

$$|p + (k+1)r, j, k; r\rangle^\sim = \begin{cases} E_{r-\frac{1}{2}}^2 \cdots E_{-\frac{3}{2}}^2 \cdot F_{r+\frac{1}{2}}^1 \cdots F_{-\frac{1}{2}}^1 |p, j, k\rangle, & r \leq 0, \\ E_{-r+\frac{1}{2}}^1 \cdots E_{-\frac{1}{2}}^1 \cdot F_{-r+\frac{3}{2}}^2 \cdots F_{\frac{1}{2}}^2 |p, j, k\rangle, & r \geq 1 \end{cases} \quad (2.9)$$

**Lemma 2.2** *The states (2.9) satisfy the generalized highest-weight conditions (2.8).*

We thus recover all the generalized highest-weight states as *extremal* states. All these extremal states are ‘connected’ to  $|p, j, k\rangle$ , and hence to each other, as described by the following simple lemma, which will be of fundamental importance however:

**Lemma 2.3** *Given a state of the form (2.9), the original highest-weight state can be reconstructed as*

$$\begin{aligned} & E_{\frac{1}{2}}^1 \cdots E_{-r-\frac{1}{2}}^1 F_{\frac{3}{2}}^2 \cdots F_{-r+\frac{1}{2}}^2 |p + (k+1)r, j, k; r\rangle^\sim \\ &= \prod_{i=r}^{-1} \left( (i(k+1) + j + p - \frac{k}{2}) \left( (i+1)(k+1) - j + p - \frac{k}{2} \right) \right), \quad r \leq 0 \\ & E_{-\frac{1}{2}}^2 \cdots E_{r-\frac{3}{2}}^2 F_{\frac{1}{2}}^1 \cdots F_{r-\frac{1}{2}}^1 |p + (k+1)r, j, k; r\rangle^\sim \\ &= \prod_{i=1}^r \left( (i-1)(k+1) + j + p - \frac{k}{2} \right) \left( i(k+1) - j + p - \frac{k}{2} \right), \quad r \geq 1 \end{aligned} \quad (2.10)$$

One can therefore travel along the integer-spaced generalized highest-weight vectors *as long as none of the factors in the above formulae vanishes*:

$$\cdots \xrightarrow{F_{-\frac{5}{2}}^1} \bullet \xrightarrow{E_{-\frac{5}{2}}^2} \times \xrightarrow{F_{-\frac{3}{2}}^1} \bullet \xrightarrow{E_{-\frac{3}{2}}^2} \times \xrightarrow{F_{-\frac{1}{2}}^1} \bullet \xrightarrow{F_{\frac{1}{2}}^2} \times \xrightarrow{E_{-\frac{1}{2}}^1} \bullet \xrightarrow{F_{-\frac{1}{2}}^2} \times \xrightarrow{E_{-\frac{3}{2}}^1} \bullet \xrightarrow{F_{-\frac{3}{2}}^2} \times \cdots \quad (2.11)$$

The solid dots represent the states (2.9), labelled by  $\theta \in \mathbb{Z}$ , positive or negative depending on whether we move right or left (while the crosses will be considered later, in Section 3.2). For some values of the parameters involved, the ‘transitivity’ among these states fails. This happens at ‘points of no return’, when the generator that does generically shift to an adjacent extremal state turns out to be an annihilator. Therefore, at the points where the transitivity breaks down, additional highest-weight-type relations are satisfied; these do overconstrain the system in the sense that they require certain relations between the parameters  $p$ ,  $j$ , and  $k$  of the highest-weight state; these relations thus guarantee the existence of a singular vector over that highest-weight state, since the breakdown of transitivity means that no operator in the representation of the algebra would map back to the highest-weight state we started with; the resulting *subrepresentation* is conveniently encoded by the respective singular vector<sup>5</sup>.

<sup>5</sup>To avoid misunderstanding, let us note once again that *not all* singular vectors follow by such a straightforward procedure, however, as we will see shortly.



In fact, as soon as one realizes that a highest-weight vector should be thought of as a whole class of equivalent but not identical, states, one also concludes that the same is obviously true for *singular* vectors. Once we have encountered in the diagram (2.11) a point where the transitivity breaks down, there would be another such *subdiagram* growing out of that point, and the respective singular vector is in fact a class of states belonging to that subdiagram. In the  $N=2$  case, where things are easier to visualize, this will be represented as in (4.9) and (4.24).

### 3 Singular vectors of the affine $sl(2|1)$ algebra at arbitrary level

We will now construct singular vectors in the  $sl(2|1)$  Verma modules. As is the case with the  $N=2$  superconformal algebra, there are singular vectors which follow immediately from the extremal vectors [67], and we will take over from the  $N=2$  algebra the name ‘charged’ for these; the remaining  $sl(2|1)$  singular vectors will be called the MFF ones, since their construction is very similar to the standard MFF formulae [50]. In what follows, we will stress the role of extremal vectors considered in the previous section.

#### 3.1 The ‘charged’ $sl(2|1)$ singular vectors

A part of the  $sl(2|1)$  singular vectors can be immediately read off from the diagram (2.11). Obviously, the ‘points of no return’ mean that we have a *submodule* in the original Verma module, and therefore such points are associated with a singular vector. From the above, we easily see that the positive- $r$  and negative- $r$  cases match to produce the following ‘special’ values of the parameters:

$$\begin{aligned} p_2(r, j, k) &= -j + \frac{1}{2}k - r(k+1), \quad r \in \mathbb{Z}, \\ p_1(r, j, k) &= j + \frac{1}{2}k - r(k+1), \quad r \in \mathbb{Z}, \end{aligned} \quad (3.1)$$

at which singular vectors occur.

The singular vectors themselves are also read off from the construction (2.9) of the extremal states:

$$|E(r, j, k)\rangle_{\text{ch}}^{(2)} = \begin{cases} \underbrace{E_{r-\frac{1}{2}}^2 \cdots E_{-\frac{3}{2}}^2}_{-r} \cdot \underbrace{F_{r+\frac{1}{2}}^1 \cdots F_{-\frac{1}{2}}^1}_{-r} |p_2(r, j, k), j, k\rangle, & r \leq -1, \\ \underbrace{E_{-r+\frac{1}{2}}^1 \cdots E_{-\frac{1}{2}}^1}_r \cdot \underbrace{F_{-r+\frac{1}{2}}^2 F_{-r+\frac{3}{2}}^2 \cdots F_{\frac{1}{2}}^2}_{r+1} |p_2(r, j, k), j, k\rangle, & r \geq 0, \end{cases} \quad r \in \mathbb{Z} \quad (3.2)$$

These will be called the charged-II singular vectors.

**Lemma 3.1** *The charged-II singular vectors  $|E(r, j, k)\rangle_{\text{ch}}^{(2)}$  satisfy the  $\theta = r$ -case of the highest-weight conditions*

$$\begin{aligned} E_{\geq -\theta+\frac{1}{2}}^1 &\approx 0, \quad E_{\geq \theta-\frac{1}{2}}^2 \approx 0, \quad E_{\geq 0}^{12} \approx 0, \quad H_{\geq 1}^+ \approx 0, \\ F_{\geq \theta+\frac{1}{2}}^1 &\approx 0, \quad F_{\geq -\theta+\frac{1}{2}}^2 \approx 0, \quad F_{\geq 1}^{12} \approx 0, \quad H_{\geq 1}^- \approx 0, \end{aligned} \quad (\theta - \frac{1}{2})k + H_0^- + H_0^+ \approx 0 \quad (3.3)$$

These special highest-weight conditions, which are stronger than (2.8), will generally be referred to as *topological*, the terminology taken over from the case of the  $N=2$  superconformal algebra. As we are

going to see, there exist two types of the topological highest-weight conditions, and, accordingly, two types of the associated singular vectors. More precisely, equations (3.3) will be called the generalized *topological-II* highest-weight conditions. The ‘ordinary’ topological-II highest-weight conditions will, as usual, be the  $\theta = 0$ -case of the generalized ones. As explained above, the appearance of the topological highest-weight conditions reflects the existence of a relation between the parameters of the highest-weight state.

We thus see that the  $r$ th charged singular vector satisfies the  $\theta = r$ -case of the generalized topological-II highest-weight conditions. This breaks down the transitivity in (2.11) and is therefore sufficient for the corresponding state to be singular. One may also wish to choose a system of representatives of singular vectors that would satisfy a single, ‘fixed’ rather than ‘relative’, set of highest-weight conditions. These highest-weight conditions cannot then be topological; instead, it is rather natural to choose the  $\theta = 0$  case of the highest-weight conditions (2.8). We then in a standard way (see (2.11)), find the  $\theta = 0$ -representative, as

$$|S(r, j, k)\rangle_{\text{ch}}^{(2)} = \begin{cases} E_{\frac{1}{2}}^1 \dots E_{-r-\frac{1}{2}}^1 F_{\frac{3}{2}}^2 \dots F_{-r-\frac{1}{2}}^2 |E(r, j, k)\rangle^{(2)}, & r \leq -1, \\ E_{-\frac{1}{2}}^2 \dots E_{r-\frac{3}{2}}^2 F_{\frac{1}{2}}^1 \dots F_{r-\frac{1}{2}}^1 |E(r, j, k)\rangle^{(2)}, & r \geq 0, \end{cases} \quad r \in \mathbb{Z}. \quad (3.4)$$

As is easy to check, these do indeed satisfy the  $\theta = 0$ -case of the highest-weight conditions (2.8).

As we see from this example, the topological representative of a singular vector is the ‘minimal’ one in that all the other representatives are obtained by making more steps along the the diagram of the type of (2.11), already *inside* the *submodule*.

Similarly, we define the charged-I singular vectors by

$$|E(r, j, k)\rangle_{\text{ch}}^{(1)} = \begin{cases} \underbrace{E_{r-\frac{1}{2}}^2 \dots E_{-\frac{3}{2}}^2}_{-r} \cdot \underbrace{F_{r-\frac{1}{2}}^1 F_{r+\frac{1}{2}}^1 \dots F_{-\frac{1}{2}}^1}_{-r+1} |p_1(r, j, k), j, k\rangle, & r \leq 0, \\ \underbrace{E_{-r+\frac{1}{2}}^1 \dots E_{-\frac{1}{2}}^1}_r \cdot \underbrace{F_{-r+\frac{3}{2}}^2 \dots F_{\frac{1}{2}}^2}_r |p_1(r, j, k), j, k\rangle, & r \geq 1, \end{cases} \quad r \in \mathbb{Z} \quad (3.5)$$

From Eqs. (2.10) we have

**Lemma 3.2** *The charged-I singular vectors (3.5) satisfy the highest-weight conditions*

$$\begin{aligned} E_{\geq -\theta+\frac{1}{2}}^1 &\approx 0, & E_{\geq \theta-\frac{1}{2}}^2 &\approx 0, & E_{\geq 0}^{12} &\approx 0, & H_{\geq 1}^+ &\approx 0, \\ F_{\geq \theta-\frac{1}{2}}^1 &\approx 0, & F_{\geq -\theta+\frac{3}{2}}^2 &\approx 0, & F_{\geq 1}^{12} &\approx 0, & H_{\geq 1}^- &\approx 0, \end{aligned} \quad (\theta - \tfrac{1}{2})k - H_0^- + H_0^+ \approx 0. \quad (3.6)$$

These will be called the *topological-I* highest-weight conditions. As before, the  $\theta = 0$ -representatives of the charged-I singular vectors follow by taking a further trip over the diagram of the type of (2.11), and read

$$|S(r, j, k)\rangle_{\text{ch}}^{(1)} = \begin{cases} E_{\frac{1}{2}}^1 \dots E_{-r-\frac{1}{2}}^1 F_{\frac{3}{2}}^2 \dots F_{-r+\frac{1}{2}}^2 |E(r, j, k)\rangle^{(1)}, & r \leq -1, \\ E_{-\frac{1}{2}}^2 \dots E_{r-\frac{3}{2}}^2 F_{\frac{1}{2}}^1 \dots F_{r-\frac{3}{2}}^1 |E(r, j, k)\rangle^{(1)}, & r \geq 0, \end{cases} \quad r \in \mathbb{Z}. \quad (3.7)$$

(and satisfy the  $\theta = 0$ -case of the highest-weight conditions (2.8)).

It is readily seen that the singular vectors (3.3) and (3.6) are related by a combination of the automorphism (2.3) and the spectral flow transform (2.4) with  $\theta = -1$ ; the same is obviously true for the vectors (3.4) and (3.7):

$$\begin{aligned} |S(r, j, k)\rangle_{\text{ch}}^{(2)} &\mapsto |S(-r, j, k)\rangle_{\text{ch}}^{(1)} \\ (E_n^1, E_n^2, F_n^1, F_n^2) &\mapsto (E_{n+1}^2, E_{n-1}^1, -F_{n-1}^2, -F_{n+1}^1) \\ H_n^+ &\mapsto -H_n^+ + k\delta_{n,0} \end{aligned} \quad (3.8)$$

The results of this subsection can be summarized as the following theorem, which we formulate explicitly even though it is essentially a reformulation of **2.1**:

**Theorem 3.3** *Whenever  $p \pm j - \frac{k}{2} = -r(k+1)$ ,  $r \in \mathbb{Z}$ , the  $sl(2|1)$  Verma module with the highest-weight vector  $|p, j, k\rangle$  has a singular vector, whose topological representative is given by (3.2) or (3.5), and the minimal-level representative, by (3.4) or (3.7) respectively.*

### 3.2 The $sl(2|1)$ MFF construction

We now proceed to the ‘continued’ part of the construction of  $sl(2|1)$  singular vectors. Their positions (as well as positions of the charged ones) are of course known from the Kač–Kazhdan determinant [40], which for the  $sl(2|1)$  algebra has recently been reviewed in [21] (see also references therein), where an important step was made by proposing an MFF-like construction for singular vectors in the case of rational  $k+1 = \frac{p}{q}$ , and classifying a number of Verma module embedding diagrams. In what follows, we will proceed for the general (complex)  $k$  and will use the continued commutation relations in order to formulate the general construction for the  $sl(2|1)$  singular vectors<sup>6</sup>. At the most fundamental level, the general MFF construction for  $sl(2|1)$  can be formulated solely in terms of the ‘continued’ products of fermionic generators, similarly to the  $N=2$  case [67]<sup>7</sup>. In this paper, however, we will not develop this consistently ‘fermionic’ approach, and will comment on it only in a remark.

So as not to interrupt the presentation later on, we begin with a simple technical observation concerning the crosses in the diagram (2.11):

**Lemma 3.4** *Unless  $p - j - \frac{1}{2}k = 0$ , any massive highest-weight state  $|p, j, k; \theta\rangle$  is equivalent to a state  $|p - \frac{1}{2}, j - \frac{1}{2}, k\rangle^{\text{mod}(1)}$  for which the modified highest-weight conditions*

$$\begin{aligned} E_{\geq -(\theta-1)+\frac{1}{2}}^1 &\approx 0, & E_{\geq (\theta-1)+\frac{1}{2}}^2 &\approx 0, & E_{\geq 0}^{12} &\approx 0, & H_{\geq 1}^+ &\approx 0, \\ F_{\geq (\theta-1)+\frac{1}{2}}^1 &\approx 0, & F_{\geq -(\theta-1)+\frac{1}{2}}^2 &\approx 0, & F_{\geq 1}^{12} &\approx 0, & H_{\geq 1}^- &\approx 0 \end{aligned} \quad (3.9)$$

*hold. Similarly, unless  $p + j - \frac{1}{2}k = 0$ , the state  $|p, j, k; \theta\rangle$  is equivalent to a state  $|p + \frac{1}{2}, j - \frac{1}{2}, k\rangle^{\text{mod}(2)}$ , which satisfies the highest-weight conditions (3.9) with  $\theta \rightarrow \theta + 1$ .*

Indeed, we build the state  $|\rangle^{\text{mod}(1)}$  as

$$\left|p - \frac{1}{2}, j - \frac{1}{2}, k; \theta\right\rangle^{\text{mod}(1)} = F_{\theta-\frac{1}{2}}^1 |p, j, k; \theta\rangle, \quad \text{then} \quad |p, j, k; \theta\rangle = \frac{1}{p-j-\frac{1}{2}k} E_{-\theta+\frac{1}{2}}^1 \left|p - \frac{1}{2}, j - \frac{1}{2}, k; \theta\right\rangle^{\text{mod}(1)}. \quad (3.10)$$

<sup>6</sup>As we have remarked, in the case of rational  $k$  one applies the MFF-like construction as many times as the rationality of  $k$  allows one to (i.e., representing the spin  $j$  as the RHSs of (3.24) with different  $r$  and  $s$ ).

<sup>7</sup>and in fact in the spirit of the suggestion of ref. [65] for (a particular representation of) the  $N=4$  algebra.

If, however,  $p - j - \frac{1}{2}k$  does vanish, then  $|p - \frac{1}{2}, j - \frac{1}{2}, k; \theta\rangle^{\text{mod}(1)}$  satisfies the topological-I highest-weight conditions and hence the massive highest-weight state cannot be recovered.

We now proceed to the generalization of the MFF construction to  $sl(2|1)$ .

### Generic case

The generalization of the MFF construction that we will propose is motivated by the following observations.

**Lemma 3.5** *The state*

$$\left| p + \frac{1}{2}, j - r - \frac{1}{2}, k, r \right\rangle'' = \underbrace{F_{\frac{1}{2}}^1 \dots F_{r-\frac{1}{2}}^1}_r \cdot \underbrace{F_{-r+\frac{1}{2}}^2 \dots F_{\frac{1}{2}}^2}_{r+1} |p, j, k\rangle$$

rewrites as

$$= (F_0^{12})^r F_{\frac{1}{2}}^2 |p, j, k\rangle, \quad r \in \mathbb{N}_0. \quad (3.11)$$

Then going back from  $|\rangle''$  to the highest-weight state  $|p, j, k\rangle$  can be achieved as

$$E_{-\frac{1}{2}}^2 \dots E_{r-\frac{1}{2}}^2 \cdot E_{-r+\frac{1}{2}}^1 \dots E_{-\frac{1}{2}}^1 \left| p + \frac{1}{2}, j - r - \frac{1}{2}, k, r \right\rangle'' = r! (j + p - \frac{k}{2}) \prod_{i=1}^r (2j - i) |p, j, k\rangle \quad (3.12)$$

if none of the factors on the right-hand side vanishes; if one of them does, this means, by the argument we have expanded above, that we encounter a singular vector at a certain point on the way. An important point is that the possibilities for this to happen are of two sorts: either  $j + p - \frac{k}{2} = 0$ , which signifies a topological point according to **3.4**, or one of the  $2j - i$  factors vanishes. We now concentrate on the basic case when this latter factor vanishes for  $i = r$  (otherwise, one should simply notice an earlier appearance of a singular vector and consider the same formulae for smaller  $r$ ). This singular vector is going to be one of the key ingredients of the MFF construction. However, the presence of the first factor on the RHS of (3.12) leads to an important subtlety, which will be discussed on page 15.

Similarly, acting first with the modes of  $F^1$  and then,  $F^2$ , as<sup>8</sup>

$$\begin{aligned} \left| p - \frac{1}{2}, j - r - \frac{1}{2}, k, r \right\rangle' &= \underbrace{F_{\frac{3}{2}}^2 \dots F_{r+\frac{1}{2}}^2}_r \cdot \underbrace{F_{-r-\frac{1}{2}}^1 \dots F_{-\frac{1}{2}}^1}_{r+1} |p, j, k\rangle, \\ &= (F_0^{12})^r F_{-\frac{1}{2}}^1 |p, j, k\rangle, \quad r \in \mathbb{N}, \end{aligned} \quad (3.13)$$

we will have

$$E_{\frac{1}{2}}^1 \dots E_{r+\frac{1}{2}}^1 \cdot E_{-r-\frac{1}{2}}^2 \dots E_{-\frac{3}{2}}^2 \left| p - \frac{1}{2}, j - r - \frac{1}{2}, k, r \right\rangle' = r! (-j + p - \frac{k}{2}) \prod_{i=1}^r (2j - i) |p, j, k\rangle, \quad (3.14)$$

The singular vector that can be read off from (3.11) or (3.13) would not yet be of the standard type, instead it would satisfy the *modified* highest-weight conditions (3.9). To obtain a singular vector that

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<sup>8</sup>Here and in what follows,  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ .

would satisfy the standard highest-weight conditions (2.8) with  $\theta = 0$ , we have to further act with the appropriate modes of  $E^1$  or  $E^2$ , which would give

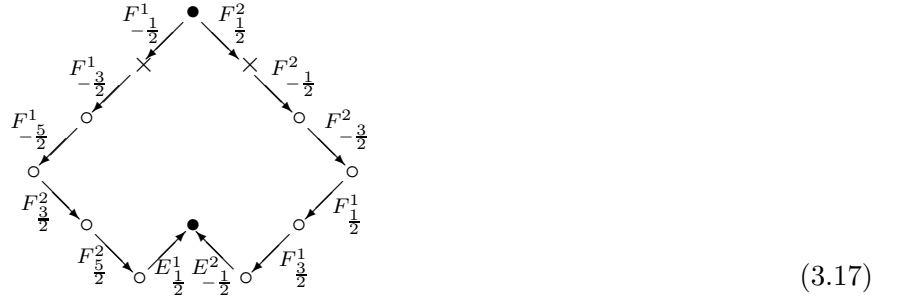
$$\begin{aligned} E_{\frac{1}{2}}^1 |p - \frac{1}{2}, j - r - \frac{1}{2}, k, r\rangle' &= \left( r(F_0^{12})^{r-1} F_{\frac{1}{2}}^2 F_{-\frac{1}{2}}^1 + (p - j - \frac{k}{2})(F_0^{12})^r \right) |p, j, k\rangle \\ E_{-\frac{1}{2}}^2 |p + \frac{1}{2}, j - r - \frac{1}{2}, k, r\rangle'' &= \left( r(F_0^{12})^{r-1} F_{\frac{1}{2}}^2 F_{-\frac{1}{2}}^1 + (p + j - r - \frac{k}{2})(F_0^{12})^r \right) |p, j, k\rangle \end{aligned} \quad (3.15)$$

**Lemma 3.6** For  $r = 2j$ , both expressions (3.15) become equal to

$$\left( 2j(F_0^{12})^{2j-1} F_{\frac{1}{2}}^2 F_{-\frac{1}{2}}^1 + (p - j - \frac{k}{2})(F_0^{12})^{2j} \right) |p, j, k\rangle, \quad (3.16)$$

which satisfies the highest-weight conditions (2.8).

In the spirit of (2.11), we now have the following ‘kite’ diagram (the  $r = 2$  example):



The construction extending (3.13) and (3.11) to  $r \leq -1$  reads

$$\begin{aligned} |p, j + r, k, r\rangle' &= \underbrace{E_{\frac{1}{2}}^2 \dots E_{-r-\frac{1}{2}}^2}_{-r} \underbrace{E_{r+\frac{1}{2}}^1 \dots E_{-\frac{1}{2}}^1}_{-r} |p, j, k\rangle \\ &= (E_{-1}^{12})^{-r} |p, j, k\rangle \end{aligned} \quad r = -1, -2, \dots \quad (3.18)$$

This satisfies the highest-weight conditions (2.8), and hence produces a singular vector, iff

$$r + 2j - k - 1 = 0. \quad (3.19)$$

Now the combination of the two ways to produce new highest-weight states out of a given one, Eqs. (3.16) and (3.18), gives the following patterns:

$$\begin{aligned} &\nearrow |p, -j, k\rangle \mapsto \dots \mapsto |p, -n(k+1) - j, k\rangle \mapsto |p, (n+1)(k+1) + j, k\rangle \mapsto \dots, \quad n \geq 0 \\ |p, j, k\rangle & \\ &\searrow |p, k+1-j, k\rangle \mapsto \dots \mapsto |p, n(k+1) - j, k\rangle \mapsto |p, -n(k+1) + j, k\rangle \mapsto \dots, \quad n \geq 1 \end{aligned} \quad (3.20)$$

As in the bosonic MFF construction [50], we now drop the condition that the corresponding  $r$  be an integer at each step in the above formulae, and require instead that only the whole sequence of mappings

lead to an element of the Verma module built on  $|p, j, k\rangle$ . Then, the resulting eigenvalue of  $H_0^-$  must differ from  $j$  by an integer (if the vector is bosonic) or a half-integer (if the vector is fermionic); from (3.16) and (3.18) we see that the former case applies, and thus the necessary conditions for the respective lines in (3.20) to yield a singular vector in the Verma module after a finite number of steps, are

$$\begin{aligned} -n(k+1) - 2j &= m, \quad \text{or} \quad (n+1)(k+1) = m, \quad n \geq 0, \quad m \in \mathbb{Z}, \\ n(k+1) - 2j &= m, \quad \text{or} \quad -n(k+1) = m, \quad n \geq 1, \quad m \in \mathbb{Z}. \end{aligned} \quad (3.21)$$

The tentative singular vectors are then constructed as the MFF monomials

$$\dots E_{\frac{1}{2}}^1 (F_0^{12})^{2(k+1)+2j} F_{-\frac{1}{2}}^1 \cdot (E_{-1}^{12})^{k+1+2j} \cdot E_{\frac{1}{2}}^1 (F_0^{12})^{2j} F_{-\frac{1}{2}}^1 |p, j, k\rangle \quad (3.22)$$

and

$$\dots (E_{-1}^{12})^{3(k+1)-2j} \cdot E_{\frac{1}{2}}^1 (F_0^{12})^{2(k+1)-2j} F_{-\frac{1}{2}}^1 \cdot (E_{-1}^{12})^{k+1-2j} |p, j, k\rangle \quad (3.23)$$

respectively, with  $j$  determined by (3.21). Further, we require that the eigenvalue of  $H_0^-$  on these states differ from  $j$  by a *positive* integer for (3.23) (in analogy with (3.18)) and a *negative* one for (3.22) (in analogy with (3.16)), which would also guarantee that the factor in the middle of the MFF monomial would always be raised to a *positive* integer power. Thus the spins in (3.22) and (3.23) are, respectively,

$$\begin{aligned} j^+(r, s, k) &= \frac{r}{2} - \frac{s-1}{2}(k+1), \\ j^-(r, s, k) &= -\frac{r}{2} + \frac{s}{2}(k+1), \end{aligned} \quad r, s \in \mathbb{N} \quad (3.24)$$

Thus the singular vectors become

$$\begin{aligned} |\text{MFF}(r, s, p, k)\rangle^+ &= E_{\frac{1}{2}}^1 (F_0^{12})^{r+(s-1)(k+1)} F_{-\frac{1}{2}}^1 \cdot (E_{-1}^{12})^{r+(s-2)(k+1)} \cdot \dots \cdot E_{\frac{1}{2}}^1 (F_0^{12})^{r-(s-1)(k+1)} F_{-\frac{1}{2}}^1 \cdot \\ &\quad \cdot \left| p, \frac{r}{2} - \frac{s-1}{2}(k+1), k \right\rangle \\ |\text{MFF}(r, s, p, k)\rangle^- &= (E_{-1}^{12})^{r+(s-1)(k+1)} \cdot E_{\frac{1}{2}}^1 (F_0^{12})^{r+(s-2)(k+1)} F_{-\frac{1}{2}}^1 \cdot \dots \cdot (E_{-1}^{12})^{r-(s-1)(k+1)} \cdot \\ &\quad \cdot \left| p, \frac{r}{2} + \frac{s}{2}(k+1), k \right\rangle \end{aligned} \quad r, s \in \mathbb{N}, \quad p \in \mathbb{C} \quad (3.25)$$

In order to give meaning to this algebraically continued construction, we need several commutation relations:

**Lemma 3.7** *For  $n \in \mathbb{N}$ , the following commutation relations hold in the universal enveloping algebra of  $sl(2|1)$ :*

$$\begin{aligned} (F_0^{12})^n E_m^{12} &= \left( -n(n-1)F_m^{12} - 2nH_m^- F_0^{12} + E_m^{12} F_0^{12} F_0^{12} \right) (F_0^{12})^{n-2}, \\ (F_0^{12})^n E_m^1 &= \left( -nF_m^2 + E_m^1 F_0^{12} \right) (F_0^{12})^{n-1}, \\ (F_0^{12})^n E_m^2 &= \left( nF_m^1 + E_m^2 F_0^{12} \right) (F_0^{12})^{n-1}, \\ (F_0^{12})^n H_m^- &= \left( nF_m^{12} + H_m^- F_0^{12} \right) (F_0^{12})^{n-1}. \end{aligned} \quad (3.26)$$

Similarly,

**Lemma 3.8** *For  $n \in \mathbb{N}$ , the following commutation relations hold:*

$$\begin{aligned}
(E_{-1}^{12})^n F_m^{12} &= \left( -n(n-1)E_{m-2}^{12} - k n \delta_{m-1,0} E_{-1}^{12} + 2n H_{m-1}^- E_{-1}^{12} + F_m^{12} E_{-1}^{12} E_{-1}^{12} \right) (E_{-1}^{12})^{n-2}, \\
(E_{-1}^{12})^n F_m^1 &= \left( n E_{m-1}^2 + F_m^1 E_{-1}^{12} \right) (E_{-1}^{12})^{n-1}, \\
(E_{-1}^{12})^n F_m^2 &= \left( -n E_{m-1}^1 + F_m^2 E_{-1}^{12} \right) (E_{-1}^{12})^{n-1}, \\
(E_{-1}^{12})^n H_m^- &= \left( -n E_{m-1}^{12} + H_m^- E_{-1}^{12} \right) (E_{-1}^{12})^{n-1}.
\end{aligned} \tag{3.27}$$

These relations are now to be extended to  $n \in \mathbb{C}$ , by first continuing to  $n \in \mathbb{Z}$  and then simply postulating them to hold for arbitrary  $n$ .

While the steps leading to (3.25) were merely a motivation, now that the formulae (3.25) are written down, we have the following

**Theorem 3.9** *For generic (complex) values of the  $U(1)$  charge  $p$  the level  $k$ , and  $r, s \in \mathbb{N}$ , the monomial expressions (3.25) determine singular vectors in the Verma modules with the highest-weight vectors  $|p, \frac{r}{2} - \frac{s-1}{2}(k+1), k\rangle$  and  $|p, \frac{-r}{2} + \frac{s}{2}(k+1), k\rangle$  respectively. For  $s \geq 2$ , the non-integral powers of  $E_{-1}^{12}$  and  $F_0^{12}$  are ‘resolved’ using the formulae of Lemmas 3.7 and 3.8: the repeated use of relations (3.27) and (3.26) in the expressions for  $|\text{MFF}(r, s, p, k)\rangle^\pm$  leads eventually to the standard ‘Verma’ form of these vectors. (No rearrangements are needed for  $s = 1$ ; ‘generic’ here refers to all values of  $p$  and  $k$  except the  $s^2 - s$  points (3.33), to be considered separately.)*

**Remark 3.10** A common feature of the (continued) *monomial* constructions for singular vectors [50, 66, 67] is that the highest-weight conditions are formally fulfilled after the application of each subsequent ‘continued’ factor, starting with the first one acting on the highest-weight state (in (3.25), these are separated with dots). However, it is only after the application of precisely as many factors as prescribed by the entire formula that the resulting vectors would belong to the Verma module.

**Remark 3.11** It is instructive (although not very useful for practical calculations) to rewrite (3.25) in a purely fermionic form, using the continued products of the fermionic generators:

$$\begin{aligned}
|\text{MFF}(r, s, p, k)\rangle^+ &= E_{\frac{1}{2}}^1 f^2\left(\frac{3}{2}, r + \frac{1}{2} + (s-1)(k+1)\right) f^1\left(-r - \frac{1}{2} - (s-1)(k+1), \frac{1}{2}\right) \\
&\quad \cdot \dots \\
&\quad \cdot e^2\left(\frac{1}{2}, -r - \frac{1}{2} + (s-2)(k+1)\right) e^1\left(r + \frac{1}{2} - (s-2)(k+1), \frac{1}{2}\right) \\
&\quad \cdot f^2\left(\frac{3}{2}, r + \frac{1}{2} - (s-1)(k+1)\right) f^1\left((s-1)(k+1) - r - \frac{1}{2}, -\frac{1}{2}\right) |p, j^+(r, s, k), k\rangle
\end{aligned} \tag{3.28}$$

and similarly for  $\text{MFF}^-$ . Here, e.g.,  $f^1(a, b) \text{ “=” } \prod_a^b F_\alpha^1$ , which can be given an algebraic meaning similarly to how this has been done for the ‘continued products’ of the  $N=2$  fermionic generators [67], as discussed in the Introduction. Such operators produce the ‘continued’ extremal states out of the chosen vacuum. Note that the grouping of the factors in the last formula is slightly different from the one that was implied when constructing (3.25): namely, the factor  $E_{\frac{1}{2}}^1 (F_0^{12})^\mu F_{-\frac{1}{2}}^1$  lends the left  $E_{\frac{1}{2}}^1$  to the  $E$ -factors further on the left in the formula.

Singular vectors in the ‘Ramond’ sector – which we will use in the next section – or in any other ‘sector’ follow by applying to the above expressions the spectral flow transform; this can be done directly in the monomial forms (3.25), and (3.2) and (3.5).

**Remark 3.12** The  $|\text{MFF}(r, s)\rangle^\pm$  singular vectors can formally be defined also for  $r = 0$ ; in that case, however, one readily finds that they are proportional to the highest-weight state, for example

$$\begin{aligned} |\text{MFF}(0, s, p, k)\rangle^+ &= a(s, p, k) |p, j^+(0, s, k), k\rangle, \\ a(s, p, k) &= \begin{cases} (p - \frac{k}{2}) \prod_{i=1}^{(s-1)/2} (p - \frac{k}{2} + i(k+1))(p - \frac{k}{2} - i(k+1)), & s \text{ even}, \\ \prod_{i=1}^{s/2} (p - \frac{k}{2} + (i - \frac{1}{2})(k+1))(p - \frac{k}{2} - (i - \frac{1}{2})(k+1)), & s \text{ odd} \end{cases} \end{aligned} \quad (3.29)$$

and similarly for  $|\text{MFF}(0, s, p, k)\rangle^-$ .

### ‘Exceptional’ points

A peculiarity of the  $s\ell(2|1)$  case, not seen in the standard MFF construction is that, essentially due to the presence of fermions, the MFF formulae might vanish at certain points in the  $p, j, k$  parameter space. That this is possible can be seen from 3.4 and the observations made before Eq. (3.15): one might happen to be unable to return from the modified highest-weight conditions to the standard ones.

Recall first of all that, as we saw in (3.15)–(3.16), the MFF singular vector can be alternatively defined with the fermions  $E^1$  and  $F^1$  replaced by  $E^2$  and  $F^2$  respectively, e.g.,

$$\begin{aligned} |\text{MFF}(r, s, p, k)\rangle^+ &= E_{-\frac{1}{2}}^2 (F_0^{12})^{r+(s-1)(k+1)} F_{\frac{1}{2}}^2 \cdot (E_{-1}^{12})^{r+(s-2)(k+1)} \cdot \dots \cdot E_{-\frac{1}{2}}^2 (F_0^{12})^{r-(s-1)(k+1)} F_{\frac{1}{2}}^2 \cdot \\ &\quad \cdot \left| p, \frac{r}{2} - \frac{s-1}{2}(k+1), k \right\rangle \end{aligned} \quad (3.30)$$

As we have remarked, dropping an arbitrary number of factors from the left of the MFF formulae does still produce a state which, while not being a Verma module element, does formally satisfy the highest-weight conditions. Let  $j_i$ ,  $i \geq 1$ , be the spin (the eigenvalue of  $H_0^-$ ) of such a state obtained by keeping  $i - 1$  factors acting on the highest-weight vector:

$$j_i = \begin{cases} \frac{r}{2} - \frac{s-i}{2}(k+1), & i \text{ odd}, \\ -\frac{r}{2} + \frac{s-i+1}{2}(k+1), & i \text{ even} \end{cases} \quad (3.31)$$

– we will thus continue with the  $\text{MFF}^+$  case, the analysis for  $\text{MFF}^-$  is completely similar.

Now, the modified highest-weight conditions (3.9) are no longer equivalent to the standard ones as soon as  $p - \frac{k}{2} \pm j = 0$ , which might indeed be the case with one of the  $j_i$ . Inside the ‘continued’ formula, however, this does not necessarily imply the vanishing of the MFF monomial, since one can then use the other expression, Eq. (3.30), for the same singular vector. A different situation occurs when the respective topological highest-weight conditions are encountered inside (3.30) as well as in the formula (3.25) for the same vector. This can happen for  $j_{i_1}$  and  $j_{i_2}$  with  $i_1$  and  $i_2$  either simultaneously odd or simultaneously



even (the case when one is odd and the other is even implies  $i_2 = i_1 + 1$  and does not lead to vanishing). Let, for definiteness,  $i_1 = 2m - 1$  and  $i_2 = 2n + 1$  where  $1 \leq m \leq n \leq s$ . This implies

$$k + 1 = \frac{r}{s - m - n}$$

and therefore the MFF monomial has the following structure (we omit the highest-weight state for brevity):

$$\dots \underbrace{E_{-\frac{1}{2}}^2 (F_0^{12})^{\frac{r(n+1-m)}{s-m-n}} F_{\frac{1}{2}}^2}_{(2n+1)\text{th}} \cdot (E_{-1}^{12})^{\frac{r(n-m)}{s-m-n}} \cdot \dots \cdot (E_{-1}^{12})^{\frac{r(m-n)}{s-m-n}} \cdot \underbrace{E_{-\frac{1}{2}}^2 (F_0^{12})^{\frac{r(m-n-1)}{s-m-n}} F_{\frac{1}{2}}^2}_{(2m-1)\text{th}} \cdot \dots \cdot E_{-\frac{1}{2}}^2 (F_0^{12})^{\frac{r(1-m-n)}{s-m-n}} F_{\frac{1}{2}}^2 \quad (3.32)$$

Here, the factors  $(2n + 1)\text{th}, \dots, (2m - 1)\text{th}$  make up the MFF ‘singular vector’  $\text{MFF}^+(0, S, p = \frac{(r+1)m+(1-r)n-s}{2(s-m-n)}, k = \frac{r}{s-m-n} - 1)$ , where

$$S = n - m + 2.$$

Such vectors evaluate as in (3.29), which does give a vanishing result for the values of the parameters that we actually have. Analyzing similarly the other cases, we arrive at

**Theorem 3.13** *The MFF singular vectors  $\text{MFF}^\pm(r, s, p, k)$ , Eqs. (3.25), vanish whenever  $(p, k) = (\mathfrak{p}(r, s, m, n), \mathfrak{k}(r, s, m, n))$ , where*

$$\begin{aligned} \mathfrak{k}(r, s, m, n) + 1 &= \frac{r}{s - m - n}, \\ \mathfrak{p}(r, s, m, n) &= \frac{(r + 1)m + (1 - r)n - s}{2(s - m - n)} \end{aligned} \quad \begin{cases} 1 \leq m \leq s, \\ 0 \leq n \leq s - 1, \\ m + n \neq s, \\ s \geq 2, \end{cases} \quad (3.33)$$

**Remark 3.14** These formulae also guarantee that the charged singular vectors of both types, I and II, exist simultaneously with one of the MFF singular vectors; indeed, with the latter chosen to be  $\text{MFF}^+(r, s)$ , the charged-I and charged-II singular vectors are labelled by integers  $1 - m$  and  $n$  respectively, as follows from the fact that Eqs. (3.33) satisfy the equations

$$\begin{aligned} p &= j + \frac{k}{2} - (1 - m)(k + 1), \\ p &= -j + \frac{k}{2} - n(k + 1), \\ j &= \frac{r}{2} - \frac{s-1}{2}(k + 1). \end{aligned} \quad (3.34)$$

(Another solution,  $p = \frac{1}{2}((s - 2n)(k + 1) - 1)$ ,  $j = \frac{1}{2}(1 - s)(k + 1)$ ,  $r = 0$ , we drop in view of (3.29). Note however that (3.33) imposes restrictions on the range of  $m$  and  $n$ , for which the MFF vectors vanish; this does in no way follow from the mere fact of a simultaneous existence of several singular vectors.)

Given (3.33), we can indeed see that the topological conditions  $p \mp j - \frac{1}{2}k = 0$ , discussed in 3.4, will indeed be satisfied by one of the ‘truncated’ MFF states, since

$$p - \frac{k}{2} - j_i = \begin{cases} \frac{r(2m-i-1)}{2(s-m-n)}, & i \text{ odd}, \\ \frac{r(i-2n-2)}{2(s-m-n)}, & i \text{ even} \end{cases} \quad p - \frac{k}{2} + j_i = \begin{cases} \frac{r(i-2n-1)}{2(s-m-n)}, & i \text{ odd}, \\ \frac{r(2m-i)}{2(s-m-n)}, & i \text{ even} \end{cases} \quad (3.35)$$

would necessarily vanish as  $i$  runs from 1 to  $2s - 1$  for the MFF factors. For a fixed pair  $(r, s)$  we thus have a set of  $s^2 - s$  ‘exceptional’ points labelled by the integers  $m$  and  $n$  in the specified range.

It is possible now to define the MFF singular vectors at these vanishing points as

$$\begin{aligned} |\text{mff}(r, s, m, n, \alpha)\rangle^\pm &= \lim_{\epsilon \rightarrow 0} \left( \frac{1}{\epsilon} |\text{MFF}(r, s, \mathbf{p}(r, s, m, n) + \epsilon \cos \alpha, \mathbf{k}(r, s, m, n) + \epsilon \sin \alpha)\rangle^\pm \right) \\ &= \left. \frac{\partial}{\partial \epsilon} (|\text{MFF}(r, s, \mathbf{p}(r, s, m, n) + \epsilon \cos \alpha, \mathbf{k}(r, s, m, n) + \epsilon \sin \alpha)\rangle^\pm) \right|_{\epsilon=0} \end{aligned} \quad (3.36)$$

The result is  $\alpha$ -dependent, and therefore, for either the  $|\rangle^+$  or the  $|\rangle^-$  case, there will in fact be *two* linearly-independent singular vectors with identical quantum numbers<sup>9</sup>.

## 4 The $s\ell(2|1) \longleftrightarrow N=2$ relation

In this section we consider how a representation of the affine  $s\ell(2|1)$  algebra can be constructed starting with the  $N=2$  superconformal algebra, and study some of its properties.

As we have mentioned, the Hamiltonian reduction of  $s\ell(2|1)$  yields the  $N=2$  superconformal algebra [16, 15, 39]. The ‘inverse’ construction, that of the  $s\ell(2|1)$  *currents* in terms of the  $N=2$  algebra currents and some free fields, has been given in [64]; however, the related construction for the highest-weight states was only outlined in [64], and we are going to consider it in more detail here. In the next section we will then analyze the correspondence between singular vectors in the  $s\ell(2|1)$  and  $N=2$  highest-weight modules.

### 4.1 The $N=2$ superconformal algebra and its highest-weight modules

In this subsection we review the properties of the  $N=2$  algebra that we will need later on. We will closely follow ref. [67]. Our analysis of  $s\ell(2|1)$  will be resumed in Section 4.2.

The  $N=2$  superconformal algebra, taken in the ‘twisted’ form [25, 69], reads

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{\epsilon}{3}m\delta_{m+n,0}, \\ [\mathcal{L}_m, \mathcal{G}_n] &= (m - n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\ [\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, & m, n \in \mathbb{Z}, \\ [\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{\epsilon}{6}(m^2 + m)\delta_{m+n,0}, \\ [\mathcal{G}_m, \mathcal{Q}_n] &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{\epsilon}{3}(m^2 + m)\delta_{m+n,0}, \end{aligned} \quad (4.1)$$

where  $c$  is the central charge;  $\mathcal{L}$ ,  $\mathcal{H}$ ,  $\mathcal{Q}$  and  $\mathcal{G}$  are called the Virasoro generators, the  $U(1)$  current, the BRST current, and the spin-2 fermionic current respectively. The spectral flow transform on the  $N=2$  algebra is given by

$$\begin{aligned} \mathcal{L}_n &\mapsto \mathcal{L}_n + \theta\mathcal{H}_n + \frac{\epsilon}{6}(\theta^2 + \theta)\delta_{n,0}, & \mathcal{H}_n &\mapsto \mathcal{H}_n + \frac{\epsilon}{3}\theta\delta_{n,0}, \\ \mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}, & \mathcal{G}_n &\mapsto \mathcal{G}_{n+\theta} \end{aligned} \quad (4.2)$$

---

<sup>9</sup>The double multiplicity of singular vectors was first observed in [23], and for the  $s\ell(2|1)$  algebra, in [21].

A Verma module over the algebra (4.1) is freely generated from a highest-weight vector  $|h, \ell, k\rangle_{N=2}$  by the generators

$$\mathcal{L}_{-m}, m \in \mathbb{N}, \quad \mathcal{H}_{-m}, m \in \mathbb{N}, \quad \mathcal{Q}_{-m}, m \in \mathbb{N}_0, \quad \mathcal{G}_{-m}, m \in \mathbb{N}, \quad (4.3)$$

while  $|h, \ell, k\rangle_{N=2}$  satisfies the following set of equations:

$$\mathcal{Q}_{\geq 1} |h, \ell, k\rangle_{N=2} = \mathcal{G}_{\geq 0} |h, \ell, k\rangle_{N=2} = \mathcal{L}_{\geq 1} |h, \ell, k\rangle_{N=2} = \mathcal{H}_{\geq 1} |h, \ell, k\rangle_{N=2} = 0 \quad (4.4)$$

$$\mathcal{H}_0 |h, \ell, k\rangle_{N=2} = h |h, \ell, k\rangle_{N=2}, \quad \mathcal{L}_0 |h, \ell, k\rangle_{N=2} = \ell |h, \ell, k\rangle_{N=2}. \quad (4.5)$$

and  $k$  parametrizes the central charge as  $c = -3 - 6k$ . The parameters  $h$  and  $\ell$  are called the  $U(1)$  charge and dimension respectively.

Just as it was the case with the  $sl(2|1)$  algebra, the  $N = 2$  superconformal algebra does allow for the construction of extremal states, and these turn out to be important in its representation theory. For  $\ell \neq 0$ , the state  $|h, \ell, k\rangle_{N=2}$  has a ‘superpartner’  $|h - 1, \ell, k; 1\rangle_{N=2} = \mathcal{Q}_0 |h, \ell, k\rangle_{N=2}$ . The state  $|h - 1, \ell, k; 1\rangle_{N=2}$  belongs to the same module because  $|h - 1, \ell, k; 1\rangle_{N=2} = 1/\ell \mathcal{G}_0 |h, \ell, k; 1\rangle_{N=2}$ . We can in fact continue acting with the modes of  $\mathcal{G}$  or  $\mathcal{Q}$ , each time with the highest of those modes that do not annihilate the state. We thus arrive at the *extremal* vectors  $\dots E_{-2}, E_{-1}, E_0, E_1, E_2, \dots$ , as

$$(4.6)$$

The commutation relations (4.1) allow us to travel over the set of the extremal vectors: up to a scalar factor, every mapping between two adjacent extremal vectors can be inverted by acting with the opposite mode of the other fermion, provided the respective scalar factor does not vanish. We will in fact relabel these states as *generalized highest-weight states*  $|h, \ell, k; \theta\rangle_{N=2}$ , which satisfy (with  $\ell$  and  $h$  measuring the eigenvalues of  $\mathcal{L}_0$  and  $\mathcal{H}_0$  respectively, although are not identical to them for  $\theta \neq 0$ , see [67])

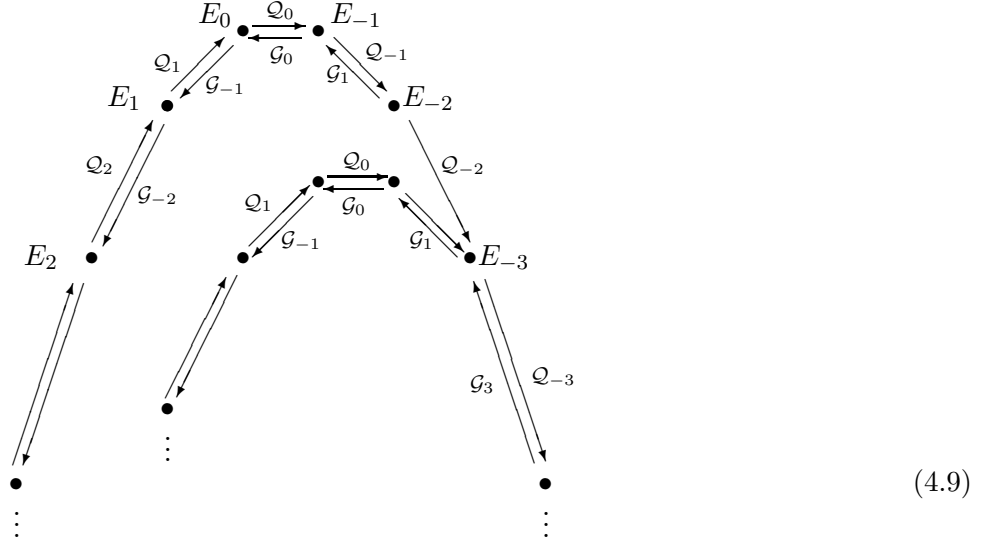
$$\begin{aligned} \mathcal{L}_m |h, \ell, k; \theta\rangle_{N=2} &= 0, \quad m \geq 1, & \mathcal{Q}_\lambda |h, \ell, k; \theta\rangle_{N=2} &= 0, \quad \lambda = -\theta + p, \quad p = 1, 2, \dots \\ \mathcal{H}_m |h, \ell, k; \theta\rangle_{N=2} &= 0, \quad m \geq 1, & \mathcal{G}_\nu |h, \ell, k; \theta\rangle_{N=2} &= 0, \quad \nu = \theta + p, \quad p = 0, 1, 2, \dots \end{aligned} \quad \theta \in \mathbb{Z} \quad (4.7)$$

The choice of a particular representative with  $\theta = 0$  in (4.4) is merely a convention as long as none of the scalar factors mentioned above vanishes. When one of them does, the corresponding extremal state  $|h, k; \theta\rangle_{\text{top}}$  satisfies stronger highest-weight conditions,

$$\begin{aligned} \mathcal{L}_m |h, k; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1, & \mathcal{Q}_\lambda |h, k; \theta\rangle_{\text{top}} &= 0, \quad \lambda \in -\theta + \mathbb{N}_0 \\ \mathcal{H}_m |h, k; \theta\rangle_{\text{top}} &= 0, \quad m \geq 1, & \mathcal{G}_\nu |h, k; \theta\rangle_{\text{top}} &= 0, \quad \nu = \theta + \mathbb{N}_0 \end{aligned} \quad \theta \in \mathbb{Z}; \quad (4.8)$$

which will be called the (generalized) *topological* highest-weight conditions (with only two parameters out of  $h, \ell, k$  remaining, since the dimension is already fixed by (4.8), see [67] for the details). The diagram

represents the case when this happens at the point  $E_{-3}$ , with  $\mathcal{G}_2 E_{-3} = 0$ :



Then we can further act upon that state with  $\mathcal{G}_1, \mathcal{G}_0, \mathcal{G}_{-1}, \dots$ , always preserving the conditions (4.7). We thus see that the extremal diagram (4.6) branches at the point where the extra conditions  $\mathcal{G}_{-r} E_{r-1} = 0$  are satisfied.

The crucial point is that the inner parabola in (4.9) corresponds to an  $N=2$  *subrepresentation*. As is elementary to show, this happens at the  $r$ th state whenever  $\ell = \ell_{\text{ch}}(r, h, k)$ , where

$$\ell_{\text{ch}}(r, h, k) = r[(r-1)(k+1) + h] \quad r \in \mathbb{Z}. \quad (4.10)$$

This is the condition for the ‘charged’ series of singular vectors to exist [18]. The corresponding singular vector  $|E(r, h, k)\rangle_{\text{ch}}$  satisfies the highest-weight conditions (4.8) with  $\theta = -r$  and is given by

$$|E(r, h, k)\rangle_{\text{ch}} = \begin{cases} \mathcal{Q}_r \dots \mathcal{Q}_0 |h, \ell_{\text{ch}}(r, h, k), k\rangle_{N=2}, & r \leq -1, \\ \mathcal{G}_{-r} \dots \mathcal{G}_{-1} |h, \ell_{\text{ch}}(r, h, k), k\rangle_{N=2}, & r \geq 1, \end{cases} \quad (4.11)$$

The minimal-level representative  $|S(r, h, k)\rangle_{\text{ch}}$  (i.e., the one at the top of the inner parabola) can be constructed as

$$|S(r, h, k)\rangle_{\text{ch}} = \begin{cases} \mathcal{G}_0 \dots \mathcal{G}_{-r-1} |E(r, h, k)\rangle_{\text{ch}}, & r \leq -1, \\ \mathcal{Q}_1 \dots \mathcal{Q}_{r-1} |E(r, h, k)\rangle_{\text{ch}}, & r \geq 1. \end{cases} \quad (4.12)$$

Its level is equal to  $|r|$  and the relative charge is  $\pm 1$  and in fact equals  $r/|r|$  (whence the name, *charged* singular vectors). It satisfies the same highest-weight conditions as those we had imposed on the highest-weight states in (4.4).

The case  $r = 0$  is somewhat special because then the topological highest-weight conditions are satisfied already at the top of the parabola (4.6). One thus considers the highest-weight vectors  $|h, k\rangle_{\text{top}}$  that satisfy the topological highest-weight conditions (4.8) for  $\theta = 0$ . The singular vectors that can be defined in modules over  $|h, k\rangle_{\text{top}}$  occur whenever the  $U(1)$  charge  $h$  of the highest-weight state is one of the following [66]:

$$\begin{aligned} \mathbf{h}^+(r, s, k) &= -(r-1)(k+1) + s - 1, \\ \mathbf{h}^-(r, s, k) &= (r+1)(k+1) - s, \end{aligned} \quad r, s \in \mathbb{N} \quad (4.13)$$

They are given in terms of continued products of the fermionic generators  $g(a, b)$  “=”  $\prod_a^b \mathcal{G}_\alpha$ , and  $q(a, b)$  “=”  $\prod_a^b \mathcal{Q}_\alpha$  [66, 67], which we simply quote here in order to be precise as to the conventions and normalizations when we compare the  $s\ell(2|1)$  singular vectors with the  $N=2$  ones:

$$|E(r, s, k)\rangle^+ = g(-r, (s-1)t-1) q(-(s-1)t, r-1-t) \dots \quad (4.14)$$

$$\begin{aligned} & g((s-2)t-r, t-1) q(-t, r-1-t(s-1)) g((s-1)t-r, -1) |h^+(r, s, k), k\rangle_{\text{top}}, \\ |E(r, s, k)\rangle^- &= q(-r, (s-1)t-1) g(-(s-1)t, r-t-1) \dots \quad (4.15) \\ & q((s-2)t-r, t-1) g(-t, r-(s-1)t-1) q((s-1)t-r, -1) |h^-(r, s, k), k\rangle_{\text{top}} \\ & r, s \in \mathbb{N} \end{aligned}$$

where

$$t \equiv \frac{1}{k+1}.$$

The  $|E(r, s, k)\rangle^\pm$  singular vectors are on level  $rs + \frac{1}{2}r(r-1)$  over the corresponding topological highest-weight state and have relative charge  $\pm r$ . They satisfy the topological highest-weight conditions (4.8) with  $\theta = \mp r$ . The algebraic rearrangement rules [66], which we briefly mentioned in the Introduction, allow one to rewrite each of these singular vectors in the ‘Verma’ form, as  $\mathcal{E}^\pm(r, s, k) |h^\pm(r, s, k), k\rangle$ , where  $\mathcal{E}^\pm(r, s, k)$  are polynomials in the usual creation operators in the module. To continue with the remark made after Theorem 3.9, the topological highest-weight conditions (4.8) are fulfilled after the application of each successive  $g$ - or  $q$ - operator; in contrast with a simpler case of Kač–Moody algebras, however, these conditions hold with a different  $\theta$  at each point.

Further, singular vectors may also exist in the modules  $U_{h,\ell,k}$  built on  $|h, \ell, k\rangle_{N=2}$ ,  $\ell \neq 0$ ; these will be called ‘massive’ singular vectors. They are given, again, by a continued construction [67] (see also [23] for a different approach). In order to fix the relative charge and the level, one can choose the highest-level representative by fixing  $\theta = 0$  in the conditions (4.7). Then the singular vectors in  $U_{h,\ell,k}$  are required to satisfy the annihilation conditions that coincide with those from (4.4). The relative charge of these representatives of massive singular vectors vanishes, and the level is equal to  $rs$ . A massive singular vector exists in  $U_{h,\ell,k}$  if  $\ell = \ell(r, s, h)$  for some  $r, s \in \mathbb{N}$ , where [18] (for  $c \neq 3$ , i.e.  $k \neq -1$ )

$$\ell(r, s, h, k) = \frac{[-(r+1)(k+1) + s + h][-(r-1)(k+1) + s - h]}{4(1+k)}, \quad r, s \in \mathbb{N} \quad (4.16)$$

Their construction can be given again in terms of ‘continued products’ of the fermionic generators  $\mathcal{G}$  and  $\mathcal{Q}$  of the algebra,  $g(a, b)$  and  $q(a, b)$  respectively:

$$\begin{aligned} |S(r, s, h, k)\rangle_{N=2} &= \mathcal{N}_1(r, s, h, k) g(0, \frac{r-3}{2} + \frac{h+s}{2(k+1)}) \\ & q(\frac{1-r}{2} - \frac{h+s}{2(k+1)}, \frac{r-1}{2} - \frac{h-s+2}{2(k+1)}) g(-\frac{r+1}{2} + \frac{h-s+2}{2(k+1)}, \frac{r-3}{2} + \frac{h+s-2}{2(k+1)}) \\ & \dots \quad (4.17) \\ & q(\frac{1-r}{2} - \frac{h-s+4}{2(k+1)}, \frac{r-1}{2} - \frac{h+s-2}{2(k+1)}) g(-\frac{r+1}{2} + \frac{h+s-2}{2(k+1)}, \frac{r-3}{2} + \frac{h-s+2}{2(k+1)}) \\ & q(\frac{1-r}{2} - \frac{h-s+2}{2(k+1)}, \frac{r-1}{2} - \frac{h+s}{2(k+1)}) g(-\frac{r+1}{2} + \frac{h+s}{2(k+1)}, -1) |h, \ell(r, s, h, k), k\rangle_{N=2} \end{aligned}$$

At the same time, this can be written in a different form, swapping the roles played by the  $q$  and  $g$  operators, as

$$\begin{aligned}
|S(r, s, h, k)\rangle_{N=2} &= \mathcal{N}_2(r, s, h, k) q\left(1, \frac{r-1}{2} - \frac{h-s}{2(k+1)}\right) \\
&g\left(-\frac{r+1}{2} + \frac{h-s}{2(k+1)}, \frac{r-3}{2} + \frac{h+s-2}{2(k+1)}\right) q\left(\frac{1-r}{2} - \frac{h+s-2}{2(k+1)}, \frac{r-1}{2} - \frac{h-s+2}{2(k+1)}\right) \\
&\dots \\
&g\left(-\frac{r+1}{2} + \frac{h+s-4}{2(k+1)}, \frac{r-3}{2} + \frac{h-s+2}{2(k+1)}\right) q\left(\frac{1-r}{2} - \frac{h-s+2}{2(k+1)}, \frac{r-1}{2} - \frac{h+s-2}{2(k+1)}\right) \\
&g\left(-\frac{r+1}{2} + \frac{h+s-2}{2(k+1)}, \frac{r-3}{2} + \frac{h-s}{2(k+1)}\right) q\left(\frac{1-r}{2} - \frac{h-s}{2(k+1)}, 0\right) |h, \ell(r, s, h, k), k\rangle_{N=2}
\end{aligned} \tag{4.18}$$

We have introduced the normalization factors

$$\mathcal{N}_1(r, s, h, k) = \prod_{n=1}^r (h - \eta^+(r, s, 1 - n, k)), \quad \mathcal{N}_2(r, s, h, k) = \prod_{n=1}^r (h - \eta^-(r, s, n, k)). \tag{4.19}$$

with

$$\eta^+(r, s, p, k) = s + (-r + 1 - 2p)(k + 1), \quad \eta^-(r, s, p, k) = -s + (r + 1 - 2p)(k + 1) \tag{4.20}$$

The above applies to the generic case, when none of the normalization factors (4.19) vanishes. When *both normalization factors (4.19) vanish*, which occurs when

$$\eta^+(r, s, -n, k) = \eta^-(r, s, m, k), \quad 0 \leq n \leq r - 1, \quad 1 \leq m \leq r, \quad m + n - r \neq 0, \quad r \geq 2 \tag{4.21}$$

that is,

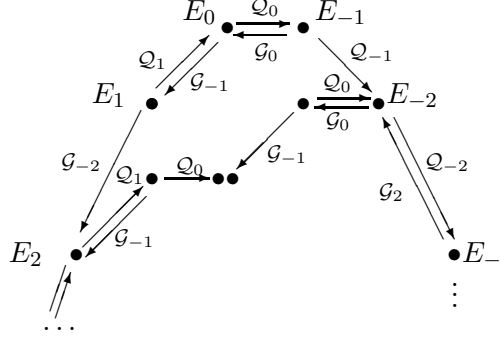
$$h = \frac{(1-m+n)s}{-m-n+r}, \quad k + 1 = \frac{s}{-m-n+r}, \tag{4.22}$$

none of the formulae (4.17), (4.18) can be applied literally, in view of the vanishing norm. In fact, while the vector  $S(r, s, h, k) \equiv \mathcal{N}_1(r, s, h, k) S^{(1)}(r, s, h, k) = \mathcal{N}_2(r, s, h, k) S^{(2)}(r, s, h, k)$  vanishes at the points (4.22), the ‘unnormalized’ vectors  $S^{(1),(2)}(r, s, \frac{(1-m+n)s}{-m-n+r}, \frac{s}{-m-n+r} - 1)$  provide a basis of a two-dimensional space of singular vectors with identical quantum numbers at each of the points (4.22). This is a preferred basis, since both vectors do then explicitly factorize through the corresponding *topological* singular vector in the centre and the products of the fermionic modes on the ends of the formulae:

$$\begin{aligned}
|\mathfrak{S}(r, s, m, n)\rangle^{(1)} &= \left( \prod_{i=0}^{-m+r-1} \mathcal{G}_i \right) \mathcal{E}^{-, -m}(r, s, \frac{s}{-m-n+r} - 1) \left( \prod_{j=-m}^{-1} \mathcal{G}_j \right) \left| \frac{(1-m+n)s}{-m-n+r}, \frac{mns}{-m-n+r}, \frac{s}{-m-n+r} - 1 \right\rangle_{N=2}, \\
|\mathfrak{S}(r, s, m, n)\rangle^{(2)} &= \left( \prod_{i=1}^{-n+r-1} \mathcal{Q}_i \right) \mathcal{E}^{+, n}(r, s, \frac{s}{-m-n+r} - 1) \left( \prod_{j=-n}^0 \mathcal{Q}_j \right) \left| \frac{(1-m+n)s}{-m-n+r}, \frac{mns}{-m-n+r}, \frac{s}{-m-n+r} - 1 \right\rangle_{N=2}
\end{aligned} \tag{4.23}$$

where  $\mathcal{E}^\pm(r, s, k)$  are the operators corresponding to the topological singular vectors (4.14), (4.15), and  $\mathcal{E}^{\pm, \theta}$  denotes the spectral flow transform of  $\mathcal{E}^\pm$ . Thus, as soon as the expressions for the topological singular vectors are known [66], the states (4.23) follow by the above, quite straightforward, construction. The positive integers  $m$  and  $n$  measure the distance along the diagram (4.6), from  $E_0$  to two branching

points, one on the left and the other on the right half of the parabola (4.6). As a simple example consider the case  $r = 2$ ,  $s = 1$ ,  $m = 2$ ,  $n = 1$  (vectors at level  $rs = 2$ ) [67]:



$$(4.24)$$

$$\begin{aligned} |\bar{5}(2, 1, 2, 1)\rangle^{(1)} &= Q_0 Q_1 G_{-2} G_{-1} |0, -2, -2\rangle_{N=2}, \\ |\bar{5}(2, 1, 2, 1)\rangle^{(2)} &= G_{-1} G_0 Q_{-1} Q_0 |0, -2, -2\rangle_{N=2} \end{aligned}$$

This completes our review of the  $N=2$  algebra and its singular vectors.

## 4.2 Constructing the $sl(2|1)$ currents

The affine  $sl(2|1)$  algebra can be constructed in terms of an arbitrary  $N=2$  superconformal matter (where, given the commutation relations (4.1), we introduce the currents  $\mathcal{T}(z) = \sum_{n \in \mathbb{N}} \mathcal{L}_n z^{-n-2}$ ,  $\mathcal{G}(z) = \sum_{n \in \mathbb{N}} \mathcal{G}_n z^{-n-2}$ ,  $\mathcal{Q}(z) = \sum_{n \in \mathbb{N}} \mathcal{Q}_n z^{-n-1}$ , and  $\mathcal{H}(z) = \sum_{n \in \mathbb{N}} \mathcal{H}_n z^{-n-1}$ ), two free bosonic currents with opposite signatures and a free fermion  $\psi \bar{\psi}$  with the operator products

$$\partial F(z) \partial F(w) = \frac{-k/2}{(z-w)^2}, \quad \partial U(z) \partial U(w) = \frac{k/2}{(z-w)^2}, \quad \psi(z) \bar{\psi}(w) = \frac{1}{z-w}. \quad (4.25)$$

Namely,

**Theorem 4.1** ([64]) *Let the  $N=2$  central charge be  $c = -3 - 6k$ ,  $k \neq 0$ , and the free fields  $\partial F$ ,  $\partial U$  and  $\psi, \bar{\psi}$  satisfy the operator products (4.25). Then the currents*

$$\begin{aligned} E^1 &= \psi e^{\frac{1}{k}(U-F)}, & E^2 &= \bar{\psi} e^{\frac{1}{k}(U-F)}, & E^{12} &= e^{\frac{2}{k}(U-F)}, \\ H^+ &= -\frac{1}{2}\mathcal{H} + \frac{1}{2}\psi \bar{\psi}, & H^- &= \partial U, \\ F^1 &= (\mathcal{G} - \bar{\psi} \partial F - \frac{1}{2}\mathcal{H} \bar{\psi} - (k + \frac{1}{2})\partial \bar{\psi}) e^{-\frac{1}{k}(U-F)}, \\ F^2 &= (\frac{1}{2}(k+1)\mathcal{Q} + \psi \partial F - \frac{1}{2}\mathcal{H} \psi + (k + \frac{1}{2})\partial \psi) e^{-\frac{1}{k}(U-F)}, \\ F^{12} &= (-\partial F \partial F - (k+1)\partial \partial F + (k+2)T_{N=0}) e^{-\frac{2}{k}(U-F)}, \end{aligned} \quad (4.26)$$

where

$$\begin{aligned} T_{N=0} &= \frac{1}{k+2} \left( (k+1)(\mathcal{T} + \frac{1}{2}\partial \mathcal{H}) + \frac{1}{4}\mathcal{H} \mathcal{H} + \mathcal{G} \psi - \frac{1}{2}(k+1)\mathcal{Q} \bar{\psi} + \frac{1}{2}\mathcal{H} \psi \bar{\psi} \right. \\ &\quad \left. + \frac{1}{4}(1+2k)\psi \partial \bar{\psi} - \frac{1}{4}(1+2k)\partial \psi \bar{\psi} \right), \end{aligned} \quad (4.27)$$

close to the affine  $sl(2|1)$  algebra of level  $k$ <sup>10</sup>. For the Sugawara energy-momentum tensor (2.5) we find then

$$T_{\text{Sug}} = \mathcal{T} + \frac{1}{2}\partial\mathcal{H} - \frac{1}{k}\partial F \partial F - \partial\partial F + \frac{1}{k}\partial U \partial U + \frac{1}{2}\partial\bar{\psi}\psi - \frac{1}{2}\bar{\psi}\partial\psi \quad (4.28)$$

Here,  $\mathcal{T} + \frac{1}{2}\partial\mathcal{H}$  is the ‘untwisted’ energy-momentum tensor with central charge  $\mathfrak{c}$ , therefore the appearance of  $\partial\mathcal{H}$  in (4.28) is due to our choice of centreless  $\mathcal{T}$ , rather than  $\mathcal{T} + \frac{1}{2}\partial\mathcal{H}$ , as the basic field (see the commutation relations (4.1)).

**Remark 4.2** If the  $\partial F$  and  $\partial U$  currents were normalized to  $\pm 1$  over the poles in (4.25), the exponents in (4.26) would acquire the factors  $\pm\sqrt{\frac{1}{2k}}$ ,  $\pm\sqrt{\frac{2}{k}}$ , and the energy-momentum tensor in (4.28) would take the canonical form. An essential point is that, in whatever normalization, the currents  $\partial U$  and  $\partial F$  have opposite signatures; the current  $\partial U - \partial F$  is *null*.

**Remark 4.3** The construction  $T_{N=0}$  can be singled out in (4.26) only for  $k \neq -2$ ; the usefulness of this object is that it is an energy-momentum tensor, i.e., satisfies the Virasoro algebra [64] (with the ‘bosonic-matter’ central charge  $13 - \frac{6}{k+2} - 6(k+2)$ ). It is clear, however, that upon substituting  $T_{N=0}$  into the expression for  $F^{12}$ , the latter is well-defined for  $k = -2$  as well (in fact,  $F^{12}$  is *determined* by  $F^1$  and  $F^2$ ).

**Remark 4.4** In the  $N=2$  non-critical string theory, one actually finds [64] a realization of  $sl(2|1)$  in terms of a slightly different field content, namely, in addition to the superconformal matter, a complex Liouville scalar  $\partial\bar{\phi}\partial\phi$  with a superpartner,  $\bar{\psi}\psi$ , and the multiplet of fermionic and bosonic ghosts  $b\bar{c}$ ,  $\eta\xi$  and  $\beta\gamma$ ,  $\tilde{\beta}\tilde{\gamma}$ . In terms of these<sup>11</sup>, the  $\partial F$  and  $\partial U$  scalars that we had above are expressed as

$$\begin{aligned} \partial F &= -\frac{1}{2}\partial\bar{\phi} - \frac{1}{2}(3+k)\partial\phi + b\bar{c} + \frac{1}{2}\beta\gamma + \frac{1}{2}\tilde{\beta}\tilde{\gamma} + \eta\xi, \\ \partial U &= -\frac{1}{2}\partial\bar{\phi} - \frac{1}{2}(3-k)\partial\phi + b\bar{c} + \frac{1}{2}\beta\gamma + \frac{1}{2}\tilde{\beta}\tilde{\gamma} + \eta\xi. \end{aligned} \quad (4.29)$$

Substituting this into (4.26) results in a realization which is also valid for  $k = 0$ ; however, forgetting about the ghosts and working with  $\partial F$  and  $\partial U$  as independent fields makes the analysis much more compact, and we will thus proceed with the representation (4.26).

In the above realization, some properties of the  $sl(2|1)$  algebra are realized rather naturally:

**Lemma 4.5** *The spectral flow transform on the  $sl(2|1)$  generators constructed as in Theorem 4.1 is realized as*

$$\begin{aligned} \mathcal{L}_n &\mapsto \mathcal{L}_n + \theta\mathcal{H}_n - (k + \frac{1}{2})(\theta^2 + \theta)\delta_{n,0}, & \mathcal{H}_n &\mapsto \mathcal{H}_n - (2k + 1)\theta\delta_{n,0}, \\ \mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}, & \mathcal{G}_n &\mapsto \mathcal{G}_{n+\theta} \\ \psi_n &\mapsto \psi_{n-\theta}, & \bar{\psi}_n &\mapsto \bar{\psi}_{n+\theta}, \end{aligned} \quad (4.30)$$

where the first two lines represent the  $N=2$  spectral flow transform.

<sup>10</sup>In (4.26), the nested normal orderings are assumed from right to left, as  $:A:BC: \dots$ . However, in order to make the formulae shorter, we write the vertex operators  $\exp(a(U - F))$  as common factors. Thus Eqs. (4.26) should be understood by multiplying every term in the parentheses with the vertex operator and introducing the normal ordering as explained.

<sup>11</sup>To be precise, the  $N=2$  string conventions are

$$b(z)c(w) = \eta(z)\xi(w) = \frac{1}{z-w}, \quad \tilde{\beta}(z)\tilde{\gamma}(w) = \beta(z)\gamma(w) = \frac{-1}{z-w}, \quad \partial\phi(z)\partial\bar{\phi}(w) = \frac{-1}{(z-w)^2}.$$



Thus the transformation is ‘localized’ in the  $N = 2$  and  $\bar{\psi}\psi$ -sectors (to recall the actual string field content,  $\psi$  is the Liouville superpartner); a similar statement is true as regards the automorphism (2.3):

**Lemma 4.6** *For  $k \neq -1$ , the automorphism (2.3) of the  $sl(2|1)$  algebra is realized for the construction of Theorem 4.1 as*

$$\begin{aligned}\mathcal{G} &\mapsto -\frac{k+1}{2}\mathcal{Q}, & \mathcal{Q} &\mapsto -\frac{2}{k+1}\mathcal{G}, \\ \mathcal{H} &\mapsto -\mathcal{H}, \\ \psi &\mapsto \bar{\psi}, & \bar{\psi} &\mapsto \psi,\end{aligned}\tag{4.31}$$

where, again, the first two lines are an involutive automorphism of the  $N = 2$  algebra.

At the same time, the construction (4.26) does obviously break the symmetry under the automorphism (2.2) by selecting one out of the two ‘triangular’ subalgebras.

The following technical observation will be quite useful in what follows:

**Lemma 4.7** *The normal-ordered products of the field operators (4.26) evaluate as*

$$\begin{aligned}E^1 F^2 &= \frac{1}{2}(1+k)\psi \mathcal{Q} - (1+k)\partial\psi \psi, \\ E^2 F^1 &= \bar{\psi} \mathcal{G} + (1+k)\partial\bar{\psi} \bar{\psi}\end{aligned}\tag{4.32}$$

### 4.3 Constructing $sl(2|1)$ highest-weight states

Now, having seen how the  $sl(2|1)$  currents arise on the non-critical  $N = 2$  string worldsheet, we can address the problem of how the  $sl(2|1)$  representation space is prepared by the string. In order to present the construction of the representation space of  $sl(2|1)$ , we choose a particular vacuum from the family of the  $\theta$ -vacua related by the spectral flow transform. For  $\theta = \frac{1}{2}$ , we will have the Ramond state

$$\left|p, j, k; \frac{1}{2}\right\rangle,\tag{4.33}$$

for which the highest-weight conditions (2.8) become

$$\begin{aligned}E_{\geq 0}^1 \left|p, j, k; \frac{1}{2}\right\rangle &= 0, & E_{\geq 0}^2 \left|p, j, k; \frac{1}{2}\right\rangle &= 0, & E_{\geq 0}^{12} \left|p, j, k; \frac{1}{2}\right\rangle &= 0, \\ F_{\geq 1}^1 \left|p, j, k; \frac{1}{2}\right\rangle &= 0, & F_{\geq 1}^2 \left|p, j, k; \frac{1}{2}\right\rangle &= 0, & F_{\geq 1}^{12} \left|p, j, k; \frac{1}{2}\right\rangle &= 0, \\ H_0^+ \left|p, j, k; \frac{1}{2}\right\rangle &= \left(p - \frac{k}{2}\right) \left|p, j, k; \frac{1}{2}\right\rangle, & H_0^- \left|p, j, k; \frac{1}{2}\right\rangle &= j \left|p, j, k; \frac{1}{2}\right\rangle\end{aligned}\tag{4.34}$$

(as before, the conditions (2.6), unaffected by the spectral flow transform, are understood.)

A Ramond vacuum satisfying the conditions (4.34) can be constructed by tensoring an appropriate  $N = 2$  highest-weight state with the free-field vacua as follows. There are three (besides the level  $k$ ) a priori parameters: the  $U(1)$  charge and the dimension of the  $N = 2$  highest-weight state  $|h, l, k\rangle$ , and the coefficient  $a$  in the vertex operator  $e^{a(U-F)}$  in the  $U-F$ -sector<sup>12</sup>. The candidates for the  $sl(2|1)$  highest-weight states thus read

$$|k - 2p, l, k\rangle_{N=2} \otimes \left|e^{a(U-F)}\right\rangle \otimes |0\rangle_{\psi\bar{\psi}};\tag{4.35}$$

<sup>12</sup>One readily sees that  $U$  and  $F$  can only appear in the vertex operator in the combination  $U - F$  in order that this vertex operator could be a part of a primary state with respect to the currents (4.26).

yet another (discrete) parameter might seem to come from the possibility of choosing different fermionic vacua [33] in the  $\bar{\psi}\psi$  theory, but this is accounted for by the  $sl(2|1)$  spectral flow transform, and thus is not relevant once we are considering the  $sl(2|1)$  highest-weight states in a fixed (Ramond) sector. The vacuum  $|0\rangle_{\psi\bar{\psi}}$  in (4.35) is defined by

$$\psi_{\geq 1} |0\rangle_{\psi\bar{\psi}} = \bar{\psi}_{\geq 0} |0\rangle_{\psi\bar{\psi}} = 0, \quad (4.36)$$

for the *integer-moded*  $\psi$  and  $\bar{\psi}$ . With some abuse of notation,  $|e^a(U-F)\rangle$  denotes the primary state that corresponds to the vertex operator  $e^a(U-F)$ .

**Theorem 4.8** *The state (4.35) satisfies the  $sl(2|1)$  highest-weight conditions (4.34) iff the  $N=2$  dimension is  $l = l(p, j, k)$ , where  $j = ka/2$  and*

$$l(p, j, k) = -\frac{(j + p - \frac{k}{2})(1 - j + \frac{k}{2} + p)}{1 + k}. \quad (4.37)$$

Thus the  $sl(2|1)$  highest-weight state in the Ramond sector reads

$$\left|p, j, k; \frac{1}{2}\right\rangle = |k - 2p, l(p, j, k), k\rangle_{N=2} \otimes \left|e^{2j/k(U-F)}\right\rangle \otimes |0\rangle_{\psi\bar{\psi}} \quad (4.38)$$

It has dimension  $\frac{j^2 - (p - \frac{k}{2})^2}{1+k}$  with respect to the Sugawara energy-momentum tensor (4.28),

$$\begin{aligned} L_0^{\text{Sug}} |k - 2p, l(p, j, k), k\rangle_{N=2} \otimes \left|e^{2j/k(U-F)}\right\rangle \otimes |0\rangle_{\psi\bar{\psi}} \\ = \frac{j^2 - (p - \frac{k}{2})^2}{1+k} |k - 2p, l(p, j, k), k\rangle_{N=2} \otimes \left|e^{2j/k(U-F)}\right\rangle \otimes |0\rangle_{\psi\bar{\psi}}, \end{aligned} \quad (4.39)$$

and therefore becomes ‘massless’ precisely when one of the topological conditions  $j = \pm(p - \frac{k}{2})$  holds (see Eqs. (3.3) and (3.6), applied to the state (4.33)).

**Remark 4.9** The spectral flow transform (2.4) with arbitrary  $\theta$  can be applied to both sides of (4.38), where on the right-hand side the  $N=2$  spectral flow transform (4.30) will be accompanied by the ‘spectral flow transform’ on the free fermions (in particular, by replacing the vacuum  $|0\rangle_{\psi\bar{\psi}}$  with the corresponding  $q$ -vacuum [33] with  $q = \theta$ ). This gives all the generalized  $sl(2|1)$  highest-weight states  $|\tilde{p}, j, k; \theta\rangle$ .

For a given dimension  $l$  of the  $N=2$  highest-weight state  $|h, l, k\rangle_{N=2}$ , and with the  $U(1)$  charge fixed as  $h = k - 2p$ , we have therefore two solutions for the  $sl(2|1)$  spin  $j$ :

$$j = \frac{1}{2} \left( k + 1 \pm \sqrt{1 + 4p + 4p^2 + 4(k + 1)l} \right). \quad (4.40)$$

Accordingly, there are in general two ways to dress an  $N=2$  highest-weight state into an  $sl(2|1)$  highest-weight state.

Now, let us assume that there exists a ‘charged’ singular vector (4.11) over the  $N=2$  state  $|h, l, k\rangle_{N=2}$ . This means that the dimension  $l$  of the  $N=2$  highest-weight state must be of the form  $l = \ell_{\text{ch}}(r, h, k)$ , with  $\ell_{\text{ch}}$  given by (4.10); setting also  $h = k - 2p$ , we will thus have the  $N=2$  state

$$|k - 2p, r[(r - 1)(k + 1) - 2p + k], k\rangle_{N=2}.$$

Then, let us dress this state into an  $sl(2|1)$  highest-weight state according to the recipe of Theorem 4.8. We thus arrive at

**Lemma 4.10** *Every  $N = 2$  highest-weight state  $|k - 2p, \ell_{\text{ch}}(r, k - 2p, k), k\rangle$ ,  $r \in \mathbb{Z}$ , can be dressed into an  $sl(2|1)$  highest-weight state  $|p, j, k; \frac{1}{2}\rangle$  in the Ramond sector in precisely two ways, namely into states of the form (4.38) with either of the two values of  $j$ :*

$$j = p - \frac{k}{2} + (1 - r)(k + 1), \quad \text{or} \quad j = -p + \frac{k}{2} + r(k + 1); \quad (4.41)$$

*expressed in terms of  $j$  considered as an independent parameter, these relations reproduce Eqs. (3.1):*

$$p = p_1(1 - r, j, k) = j + \frac{k}{2} + (r - 1)(k + 1), \quad \text{or} \quad p = p_2(-r, j, k) = -j + \frac{k}{2} + r(k + 1) \quad (4.42)$$

*respectively. Each of the resulting  $sl(2|1)$  highest-weight states is therefore such that a charged singular vector exists on it.*

We can thus expect that the charged  $sl(2|1)$  singular vectors (3.2) and (3.5) would be related to the charged  $N = 2$  singular vectors (4.11). As is clear from the above, the existence of spectral flow transforms for each of the algebras involved allows us to consider this only for, say, charged-II  $sl(2|1)$  singular vectors (recall that the charged-I vectors follow by a combination of the spectral flow transform and the automorphism, *both of which allow for a restriction to the  $N = 2$  superconformal algebra*).

Further, let us take the  $N = 2$  highest-weight state on which a massive  $N = 2$  singular vector exists; this determines the  $\mathcal{L}_0$ -dimension as in (4.16), and thus the  $N = 2$  state under consideration is  $|k - 2p, \ell(r, s, k - 2p, k), k\rangle_{N=2}$ .

**Lemma 4.11** *Every  $N = 2$  highest-weight state  $|k - 2p, \ell(r, s, k - 2p, k), k\rangle$ ,  $r, s \in \mathbb{N}$ , can be dressed into an  $sl(2|1)$  highest-weight state in the Ramond sector in precisely two ways, namely into states of the form (4.38) with either of the two values of  $j$ :*

$$j = j^-(s, r + 1, k) = -\frac{1}{2}s + \frac{1}{2}(r + 1)(k + 1), \quad j = j^+(s, r, k) = \frac{1}{2}s - \frac{1}{2}(r - 1)(k + 1) \quad (4.43)$$

*(cf. (3.24)). Each of the resulting  $sl(2|1)$  highest-weight states is therefore such that an MFF singular vector exists on it.*

As we had  $r, s = 1, 2, \dots$ , in (4.16), we thus reproduce in (4.43) all the cases from (3.24) except  $j^-(n, 1, k)$ ,  $n \geq 1$ . These  $sl(2|1)$  highest-weight states do *not* therefore allow for a construction in terms of  $N = 2$  highest-weight states on which an  $N = 2$  singular vector can live<sup>13</sup>.

## 5 $sl(2|1)$ singular vectors on the $N = 2$ string worldsheet

As we have seen in the previous section, every  $N = 2$  Verma module  $U_{h, \ell, k}$  can be dressed, by tensoring it with free-field modules, into two  $sl(2|1)$  modules. Further, whenever  $U_{h, \ell, k}$  has a singular vector, the resulting  $sl(2|1)$  module would also have a singular vector. This applies in fact to the entire embedding diagrams of  $N = 2$  singular vectors, since the  $N = 2$  singular vectors, viewed as highest-weight states, can again be dressed according to the recipe of Theorem 4.8:

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<sup>13</sup>This fact suggests that the  $sl(2|1)$  algebra would have an additional series of fusions as compared to the  $N = 2$  algebra (similarly to an ‘extra’ series of fusions that exist in the  $sl(2)$  theory as compared to the minimal models [4, 3, 59]).

**Lemma 5.1** *Every  $N=2$  singular vector can be dressed as specified in Theorem 4.8, into a state that satisfies the  $sl(2|1)$  highest-weight conditions.*

Note that this lemma does not tell us anything about whether a given  $sl(2|1)$  singular vector can be arrived at by dressing an  $N=2$  singular vector, nor in fact whether the  $sl(2|1)$  highest-weight state resulting from the dressing would be an  $sl(2|1)$ -descendant of the chosen highest-weight state in the module.

To see that the above statement is true, we consider first the charged singular vectors. As regards their highest-weight properties, the states (4.12) can be written as  $|h + r/|r|, \ell_{\text{ch}}(r, h, k) + |r|/r, k\rangle_{N=2}$ , where  $\ell_{\text{ch}}(r, h, k) + |r|/r = r[(r-1)(k+1) + h + r/|r|]$ , hence Theorem 4.8 applies with  $k-2p \equiv h \rightsquigarrow h + r/|r|$ . As to the massive  $N=2$  singular vectors, similarly, we can write them as  $|h, \ell(r, s, h, k) + rs, k\rangle_{N=2}$ , where the dimension factorizes as

$$\ell(r, s, h, k) + rs = \frac{[s + r(k+1) - h + (k+1)][s + r(k+1) + h - (k+1)]}{4(k+1)},$$

therefore Theorem 4.8 would apply again and result in a shift of  $j$ . In fact, in both cases the effective shifts of the resulting  $sl(2|1)$  parameters are such as they would be if the resulting  $sl(2|1)$  state was the respective (charged, or MFF-)  $sl(2|1)$  singular vector. As we are going to see, this is indeed the case!

## 5.1 The charged singular vectors

Now, we proceed to a more complicated problem of a direct evaluation of  $sl(2|1)$  singular vectors in the  $N=2$  terms. Consider first the charged singular vectors. We will take them in the ‘Ramond’ sector: denote

$$|E(r, j, k)\rangle_{\text{ch}}^{(i), (R)} = \mathcal{U}_{\frac{1}{2}} |E(r, j, k)\rangle_{\text{ch}}^{(i)}, \quad (5.1)$$

with  $\mathcal{U}_{\theta}$  being the spectral flow transform operator (2.4).

Clearly, it suffices to evaluate in the realization (4.26), (4.38) the *topological representatives* of the charged  $sl(2|1)$  singular vectors (3.2), (3.5). These become, again, the topological representatives of the charged  $N=2$  singular vectors (4.11):

**Theorem 5.2** *The charged-II singular vectors (3.2), mapped into the Ramond sector, evaluate in the realization (4.26), (4.38) as the charged  $N=2$  singular vectors (4.11) tensored with the free-field vacua:*

$$|E(r, j, k)\rangle_{\text{ch}}^{(2), (R)} = \begin{cases} \mathcal{G}_r \dots \mathcal{G}_{-1} |k - 2\mathbf{p}_2(r, j, k), \ell_{\text{ch}}(-r, k - 2\mathbf{p}_2(r, j, k), k), k\rangle_{N=2} \\ \quad \otimes |e^{2j/k(U-F)}\rangle \otimes \bar{\psi}_r \dots \bar{\psi}_{-1} |0\rangle_{\psi\bar{\psi}}, & r \leq -1, \\ (-1)^r \left(\frac{k+1}{2}\right)^{r+1} \mathcal{Q}_{-r} \dots \mathcal{Q}_0 |k - 2\mathbf{p}_2(r, j, k), \ell_{\text{ch}}(-r, k - 2\mathbf{p}_2(r, j, k), k), k\rangle_{N=2} \\ \quad \otimes |e^{2j/k(U-F)}\rangle \otimes \psi_{-r+1} \dots \psi_0 |0\rangle_{\psi\bar{\psi}}, & r \geq 0 \end{cases} \quad (5.2)$$

(Of course,  $\ell_{\text{ch}}(-r, k - 2\mathbf{p}_2(r, j, k), k) = l(\mathbf{p}_2(r, j, k), j, k)$ ). Here,  $\bar{\psi}_r \dots \bar{\psi}_{-1} |0\rangle_{\psi\bar{\psi}}$  and  $\psi_{-r+1} \dots \psi_0 |0\rangle_{\psi\bar{\psi}}$  represent the ‘ $r$ ’ vacua in the  $\psi\bar{\psi}$  theory [33].

Either applying the spectral flow transform, or directly evaluating the charged-I  $sl(2|1)$  singular vectors (3.5), we arrive at

**Theorem 5.2'** *The charged-I singular vectors (3.2) mapped into the Ramond sector evaluate in the realization (4.26), (4.38) as the following tensor products of the  $N=2$  charged singular vectors with the free-field vacua::*

$$|E(r, j, k)\rangle_{\text{ch}}^{(1), (R)} = \begin{cases} (-1)^r \mathcal{G}_{r-1} \dots \mathcal{G}_{-1} |-2\mathbf{p}_1(r, j, k), \ell_{\text{ch}}(1-r, k-2\mathbf{p}_1(r, j, k), k)\rangle_{N=2} \\ \quad \otimes |e^{2j/k(U-F)}\rangle \otimes \bar{\psi}_r \dots \bar{\psi}_{-1} |0\rangle_{\psi\bar{\psi}}, & r \leq 0, \\ \left(\frac{k+1}{2}\right)^r \mathcal{Q}_{-r+1} \dots \mathcal{Q}_0 |-2\mathbf{p}_1(r, j, k), \ell_{\text{ch}}(1-r, -2\mathbf{p}_1(r, j, k), k)\rangle_{N=2} \\ \quad \otimes |e^{2j/k(U-F)}\rangle \otimes \psi_{-r+1} \dots \psi_0 |0\rangle_{\psi\bar{\psi}}, & r \geq 1 \end{cases} \quad (5.3)$$

In each case we thus get a charged  $N=2$  singular vector tensored with a  $UF$ -vertex operator and an  $|r\rangle$ -vacuum in the  $\bar{\psi}\psi$ -theory.

**Remark 5.3** Taking the fields to be literally those of the non-critical  $N=2$  string in the conformal gauge (see (4.29)), we would have  $e^{2j/k(U-F)} = e^{2j\phi}$ , and we thus see that, i) the ghosts decouple altogether (the RHSs of (5.3) and similar formulae would then contain only the product of bare ghost vacua), ii) the super-Liouville sector, on the other hand, is sensitive to which singular vector is being evaluated: it contributes  $|e^{2j\phi}\rangle \otimes |r\rangle_{\psi\bar{\psi}}$  to the  $|E(r, j, k)\rangle_{\text{ch}}$  singular vector.

Being interested in the  $N=2$  piece, and thus dropping down the  $UF$  and  $\psi\bar{\psi}$  sectors, we can summarize the situation as the following *reductions* of singular vectors:

$$\begin{array}{ccc} E(r, j, k)_{\text{ch}}^{(2)} & & E(r+1, j, k)_{\text{ch}}^{(1)} \\ & \searrow & \swarrow \\ & E(-r, k-2\mathbf{p}_2(-r, j, k), k)_{\text{ch}} & \end{array} \quad (5.4)$$

One may wish to choose different representatives for the singular vectors involved in these reductions. Such a choice would not, of course, change the fact that the charged singular vectors of the two algebras are in the  $2:1$  correspondence; a simple analysis shows that the singular vectors (3.4) and (3.7) correspond precisely to the  $N=2$  singular vectors (4.12).

It should be observed (in fact, this underlies the proof of the Theorem) that, *when evaluating the extremal vectors*, the fermionic  $sl(2|1)$  currents (4.26) behave in accordance with the effective replacements

$$\begin{aligned} E^1 &= \psi e^{\frac{1}{k}(U-F)} && \rightarrow \psi \boxtimes e^{\frac{1}{k}(U-F)}, \\ F^2 &= \left(\frac{1}{2}(k+1)\mathcal{Q} + \psi \partial F - \frac{1}{2}\mathcal{H}\psi + (k+\frac{1}{2})\partial\psi\right) e^{-\frac{1}{k}(U-F)} && \rightarrow \frac{1}{2}(k+1)\mathcal{Q} \boxtimes e^{-\frac{1}{k}(U-F)}, \\ E^2 &= \bar{\psi} e^{\frac{1}{k}(U-F)} && \rightarrow \bar{\psi} \boxtimes e^{\frac{1}{k}(U-F)}, \\ F^1 &= (\mathcal{G} - \bar{\psi} \partial F - \frac{1}{2}\mathcal{H}\bar{\psi} - (k+\frac{1}{2})\partial\bar{\psi}) e^{-\frac{1}{k}(U-F)} && \rightarrow \mathcal{G} \boxtimes e^{-\frac{1}{k}(U-F)}, \end{aligned} \quad (5.5)$$

where  $\boxtimes$  stresses the fact that the two factors are decoupled, whence  $E^1 F^2 \rightarrow \frac{1}{2}(k+1)\psi \mathcal{Q}$ ,  $E^2 F^1 \rightarrow \bar{\psi} \mathcal{Q}$ . In the extremal vectors, these generators are indeed encountered in combinations  $E_{-n}^2 F_{-n+1}^1$  or  $E_{-n}^1 F_{-n+1}^2$ , and it is to these combinations that the above replacement effectively applies. This is not surprising in view of (4.32), where the right-hand sides are in an obvious correspondence with the pairwise products of the right-hand sides from (5.5), modulo the terms that *vanish inside the extremal vectors*. Indeed, the extremal vectors in the  $\psi\bar{\psi}$ -sector, for example, rewrite in terms of the normal products as  $\partial^N \psi \dots \partial \psi \psi(z)$ , and therefore, e.g.,  $\partial \psi (\psi \mathcal{Q} + \alpha \partial \psi \psi) = \partial \psi \psi \mathcal{Q}$ .

## 5.2 The MFF vs. ‘massive’ singular vectors

Now we turn to the MFF singular vectors (3.25). Recall Lemma 4.11; we are going to formulate the ‘inverse’ statement. As before,  $^{(R)}$  will refer to the Ramond sector.

**Theorem 5.4** *The singular vectors  $|\text{MFF}^-(r, 1, p, k)\rangle$ , evaluated in the realization (4.26), (4.38), become*

$$|\text{MFF}^-(r, 1, p, k)\rangle^{(R)} = \left| k - 2p, \frac{1}{k+1} \left( \frac{r-1}{2} - p \right) \left( \frac{r+1}{2} + p \right), k \right\rangle_{N=2} \otimes \left| e^{\frac{r+k+1}{k} (U-F)} \right\rangle \otimes |0\rangle_{\psi\bar{\psi}} \quad r \geq 1. \quad (5.6)$$

In terms of the  $N=2$  algebra, therefore, these  $sl(2|1)$  singular vectors reduce to highest-weight states, *not* to a singular vector; the  $U-F$ - and  $\psi\bar{\psi}$ -sectors do decouple however.

Now, as to the remaining MFF singular vectors, we do not have a direct proof, yet on the basis of various consistency checks and explicit evaluations we formulate the following

**Theorem 5.5** (CONJECTURED)

*I. The other MFF singular vectors (3.25) evaluate as the massive  $N=2$  singular vectors tensored with free-field primary states:*

$$\begin{aligned} |\text{MFF}^-(r, s+1, p, k)\rangle^{(R)} &= (-1)^s 2^{r-s} r \left( \frac{k+1}{2} \right)^{rs} |S(s, r, k-2p, k)\rangle_{N=2} \otimes \left| e^{\frac{r+s(k+1)}{k} (U-F)} \right\rangle \otimes |0\rangle_{\psi\bar{\psi}} \\ |\text{MFF}^+(r, s, p, k)\rangle^{(R)} &= (-1)^{s-1} 2^{r+1-s} r \left( \frac{k+1}{2} \right)^{r(s-1)} \\ &\quad \times |S(s, r, k-2p, k)\rangle_{N=2} \otimes \left| e^{\frac{-r-(s-1)(k+1)}{k} (U-F)} \right\rangle \otimes |0\rangle_{\psi\bar{\psi}} \\ &\quad r, s \geq 1. \end{aligned} \quad (5.7)$$

Again, in terms of the fields on the  $N=2$  string worldsheet, it is the Liouville dressing of an  $N=2$  highest-weight state that is sensitive to which  $sl(2|1)$  singular vector is taken, while the ghosts contribute only the bare vacua.

Thus, with the exception of  $\text{MFF}^-(r, 1, p, k)$ , the MFF singular vectors (3.25) are a ‘double-covering’ of the massive  $N=2$  singular vectors. This can be summarized as follows:

$$\begin{array}{ccc} \text{MFF}^-(r, 1, p, k) & \text{MFF}^-(r, s+1, p, k) & \text{MFF}^+(r, s, p, k) \\ r \geq 1 & r, s \geq 1 & r, s \geq 1 \\ \downarrow & \searrow & \swarrow \\ \bullet & & S(s, r, k-2p, k) \end{array} \quad (5.8)$$

The Theorem is conjectured on the basis of ‘numerology’ (matching the quantum numbers), Lemma 4.11, several consistency checks and explicit evaluation in the following cases:

$$\begin{array}{ccccccccc}
r \setminus^{s+1} & 2 & & 3 & & 4 & & r \setminus^s & 1 & 2 & 3 & 4 & (5.9) \\
1 & 8/3_1 & & 36/9_2 & & 139/22_3 & & 1 & 2/3_0 & 11/9_1 & 48/22_2 & 171/51_3 \\
2 & 33/9_2 & & 442/51_4 & & & & 2 & 2/9_0 & 49/51_2 & & \\
3 & 107/22_3 & & & & & & 3 & 2/22_0 & & & \\
& & & & & & & 4 & 2/51_0 & & & 
\end{array}$$

for  $\text{MFF}^-$  and  $\text{MFF}^+$  respectively, where the notation  $m/n$  indicates that the corresponding  $sl(2|1)$  singular vector contains  $m$  terms when rewritten in the Verma form, while the respective  $N=2$  singular vector has  $n$  terms, and the subscript indicates the *level* of the  $sl(2|1)$  singular vector. The next check would be the reduction of  $|\text{MFF}(2, 3, p, k)\rangle^+$  to  $|S(3, 2, k - 2p, k)\rangle_{N=2}$ , with  $588/221_4$ , and that of  $|\text{MFF}(3, 2, p, k)\rangle^+$  to  $|S(2, 3, k - 2p, k)\rangle_{N=2}$ , with  $161/221_3$ , but these reductions seem to be too complicated to be performed explicitly.

**Remark 5.6** The  $sl(2|1)$  and  $N=2$  singular vectors that are being compared in (5.7), are initially given each in its own monomial form, and yet the respective normalizations differ only by inessential numerical factors and powers of  $(k+1)$ , whose origin is clear from (4.26). Thus no  $p$ -dependent factors or other  $k$ -dependent factors appear when the  $N=2$  singular vectors are taken directly from [67].

The ‘transposition’ of  $r$  and  $s$  observed in (5.8) does also show up in the ‘exceptional’ cases, when there are two linearly independent singular vectors with identical quantum numbers. Recall that, for both the algebras involved, these ‘special’ singular vectors have been defined by essentially ‘resolving’ a zero of the general formula. Comparing (3.33) and (4.22) we see that, indeed, the sets of ‘exceptional’ points are mapped onto each other by  $r \leftrightarrow s$ . Substituting the respective parameters into (5.7), we would have  $0 = 0$ , since the parameters (4.22) are indeed recovered as  $h \Big|_{(4.22)} = 2k(s, r, m, n) - p(s, r, m, n)$ ,  $k \Big|_{(4.22)} + 1 = k(s, r, m, n) + 1$ ; as before, we are interested in what is ‘behind’ these zeroes. We thus have the second part of the theorem:

**Theorem 5.6**

*II. For either  $|\text{mff}(r, s, m, n, \alpha)\rangle^+$ ,  $r \geq 1, s \geq 2$ , or  $|\text{mff}(r, s + 1, m, n, \alpha)\rangle^-$ ,  $r, s \geq 1$ , the two-dimensional space of  $sl(2|1)$  singular vectors at the points (3.33), defined by Eqs. (3.36), reduces to the two-dimensional space spanned by the  $N=2$  singular vectors  $|s(s, r, m, n)\rangle^{(1)}$  and  $|s(s, r, m, n)\rangle^{(2)}$  given by (4.23).*

The nature of the correspondence between the  $sl(2|1)$  and  $N=2$  singular vectors is such that it extends to the cases when one singular vector can be constructed on another one, and moreover, a ‘synchronous’ appearance of fermions in the expressions for singular vectors of the two algebras indicates that these ‘composite’ singular vectors would also vanish simultaneously. It should also be stressed that, as we see, the  $sl(2|1)$  singular vectors evaluated in the ‘ $N=2$ -string’ realization *do not* coincide; it is only upon projecting out the  $U$ - $F$  and  $\bar{\psi}\psi$ -sectors<sup>14</sup> that, in terms of the  $N=2$  algebra alone,  $\text{MFF}^-(r, s + 1, p, k)$

<sup>14</sup>In ‘stringy’ terms, the Liouville sector.

and  $\text{MFF}^+(r, s, p, k)$ , and similarly  $|E(r, j, k)\rangle_{\text{ch}}^{(2)}$  and  $|E(r+1, j, k)\rangle_{\text{ch}}^{(2)}$ , become identical. Also, none of the  $sl(2|1)$  singular vectors vanishes when evaluated in the  $N=2$  string realization of  $sl(2|1)$ , which is a crucial difference from *free*-field constructions, which usually lead to the vanishing of a number of singular vectors.

On the other hand, the above results do also apply to the Wakimoto representation of  $sl(2|1)$  [16], or more precisely, *two* Wakimoto representations [20], associated with two Weyl-inequivalent simple root systems. As shown in [64], the free-field ingredients of each of these Wakimoto representations can be constructed from the fields we had in (4.26), *once the  $N=2$  matter is ‘bosonized’ in terms of free fields*. Therefore the Wakimoto bosonizations can be *mapped through* the ‘stringy’ representation,<sup>15</sup> and thus the  $sl(2|1)$  singular vectors in the Wakimoto representations become simply the respective ‘bosonizations’ of the  $N=2$  superconformal singular vectors.

## 6 Conclusions

We have seen that representation theories of the affine  $sl(2|1)$  and  $N=2$  superconformal algebras are closely related, and have constructed explicit mappings that implement this relation. In particular, we have found how the general constructions for the affine  $sl(2|1)$  and  $N=2$  superconformal singular vectors are mapped into each other. This has been done using the representation of the affine  $sl(2|1)$  algebra realized in the non-critical  $N=2$  string. As to the general construction of singular vectors, we have seen that both the  $sl(2|1)$  and  $N=2$  singular vectors are initially given in *monomial* forms, in terms of ‘continued’ objects. It would be extremely interesting to directly map these monomial forms into each other. This is in fact what we have done for the charged series of singular vectors, but the monomials in that case did not require a continuation. We believe that in the ‘massive’ case as well, the correspondence between the  $sl(2|1)$  and  $N=2$  singular vectors exists, at the most fundamental level, for the respective ‘continued monomials’, since they in fact represent the continuation of the extremal states, which encode a significant part of the structure of the algebra and its representations [29].

At the same time, the  $N=2$  singular vectors can be mapped into the affine  $sl(2)$  singular vectors; to be more precise, the subset of topological  $N=2$  singular vectors is *isomorphic* to the standard  $sl(2)$  singular vectors, while the massive  $N=2$  singular vectors are mapped into  $sl(2)$  modules without a unique highest-weight vector [30]. It would be interesting to build a direct relation between the  $sl(2|1)$  and  $sl(2)$  singular vectors, and in particular to see how these more general  $sl(2)$  ‘Verma’ modules (those with infinitely-many equivalent ‘almost-highest-weight’ vectors) can be derived from the  $sl(2|1)$  Verma modules.

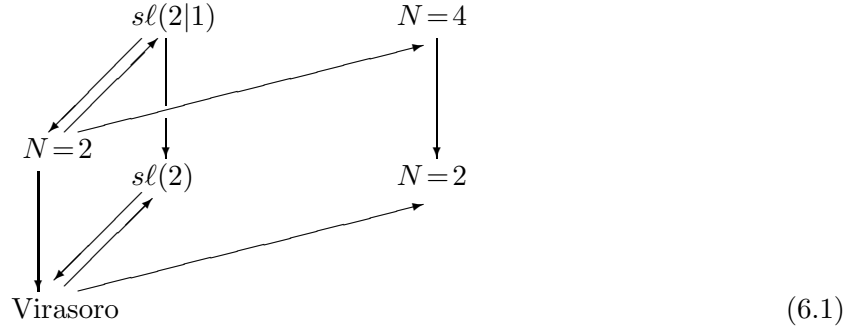
Returning to the role of the  $sl(2|1)$  representation theory in string theory, it may indeed be the case that, modulo several exceptions,<sup>16</sup> the diagram (1.2) for the singular vectors will be a ‘double-covering’

<sup>15</sup>Note, in particular, that it follows immediately from the formulae of [64] that each of the two Wakimoto bosonizations allows a natural realization of the spectral flow transform.

<sup>16</sup>i.e., several series of singular vectors, which do *not* reduce; such series are expected to exist on the general ground that ‘larger’ algebras should have extra series in their fusion rules, and this is precisely what we have seen in the  $sl(2|1) \rightarrow N=2$  reduction of singular vectors, and also what is the case for the  $N=2 \rightarrow$  Virasoro reduction.



of the diagram (1.1):



(where an arrow  $\mathcal{A} \rightarrow \mathcal{B}$  means ‘take generators of  $\mathcal{A}$  and construct generators of  $\mathcal{B}$ , possibly (for the upward arrows) using some other fields’; of course, the standard notation would be the embeddings of subalgebras, e.g.,  $sl(2) \hookrightarrow sl(2|1)$ ). The  $N=2$  algebra enters this diagram twice, once as an ‘elementary’  $N=2$  matter theory, and the other time as the algebra realized on the bosonic string worldsheet. Accordingly, the affine  $sl(2|1)$  algebra does also ‘cover’ the non-critical bosonic string [15]. As regards the proposal to define non-critical string theories as the Hamiltonian reduction of the appropriate affine (super)algebras [15, 46, 60], one needs to know how much of the physical content of the worldsheet formulation of a string theory comes with the Hamiltonian reduction, or at least which representations of the respective affine superalgebra should be taken in order to arrive at the string space of states. The present paper offers a partial result in that direction, by showing how the highest-weight states are related and also finding out which of the  $sl(2|1)$  singular vectors do, and which do not, reduce to those of the  $N=2$  superconformal matter theory. There are also some indications that a similar treatment can be applied to the  $N=4$  superconformal algebra, its highest-weight and extremal states and singular vectors.

Now that we have seen that the structure of both the  $sl(2|1)$  and  $N=2$  singular vectors reflects the existence of extremal vectors of these algebras, an interesting point is how the Lian–Zuckerman states can also be understood in terms of extremal vectors. It would also be very interesting to translate the general constructions and the reductions of singular vectors into the information about the fusion rules.

**Acknowledgements** It is a pleasure to thank D. Lüst for hospitality at Humboldt-Universität zu Berlin, where this paper was written. I am also grateful to O. Andreev, S. Ketov, C. Preitschopf, and I. Tipunin for interesting discussions. This work is supported in part by Deutsche Forschungsgemeinschaft under contract 436 RUS 113-29, and by the RFFI grant 96-01-00725.

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